Stochastic Integral Equations with Respect to Semimartingales

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To Professor K. D. Elworthy and Mrs. S. M. Elworthy

Contents

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SUMMARY

Stochastic integral equations were first developed by mathematicians as a tool for the explicit construction of the paths of diffusion processes for given coefficients of drift and diffusion. Since many physical, engineering, biological as well as social phenomena can be modelled by stochastic integral equations, the theory of stochastic integral equations has become one of the most active fields of mathematical research. This thesis considers stochastic integral equations with respect to semimartingales, which in some sense forms the most general case.

This thesis consists of five chapters. In Chapter I we first develop the theory of existence and uniqueness of solutions to stochastic integral equations with respect to semimartingales (SIES) and delay SIES. Chapter II presents the explicit representation of the solutions to linear SIES. Chapter III contains the theory of stochastic stability and boundedness. Chapter IV is for comparison theorems. Chapter V is devoted to the transformation formula which transforms stochastic integrals with respect to continuous local martingales into classical Itô's integrals with respect to Brownian motion. This formula is then applied to study properties of stochastic integrals, SIES and stability.

This thesis is mainly based on my nine new papers published during my Ph.D. study period of 1st October 1987-1st October 1989 at Warwick.

INTRODUCTION

Stochastic integral equations were first developed by mathematicians as a tool for the explicit construction of the paths of diffusion processes for given coefficients of drift and diffusion. Since many physical, engineering, biological as well as social phenomena can be modelled by stochastic integral equations, the theory of stochastic integral equations has become one of the most active fields of mathematical research. There are now several important monographs covering the field with varying emphases. For example, Arnold [1], Elliott [1], Elworthy [1], Friedman [1], Karatzas and Shreve [1], McKean [1], Rogers and Williams [1] and so on. This thesis considers stochastic integral equations with respect to semimartingales (SIES), which in some sense forms the most general case.

This thesis is divided into five chapters.

In Chapter I we first develop the theory of existence and uniqueness of solutions to SIES. In fact, as early as in 1983 we published paper " Existence and Uniqueness of the Solutions of Stochastic Differential Equations " in Stochastics which dealt with the following one-dimensional stochastic equation

$$X = \Phi(X) + F(X).M$$

where X was a real valued stochastic process and M was a semimartingale. It was Professor Elworthy who suggested that I study more general n-dimensional functional SIES

$$X = \Phi(X) + \sum_{i=1}^{m} F_i(X).M_i$$

where X was a R^n-valued stochastic process and M_i ($1 \le i \le m$) were semimartingales, so that this thesis could be developed in more general case. The results in Sections 1.3-1.5 are successfully used to study the existence and uniqueness of the solutions to delay SIES in Sections 1.6 and 1.7 which was presented at the 17th International Conference on Stochastic Processes and Their Applications, 27 June - 1 July 1988, Roma, Italy and was published in Stochastic Analysis and Applications, March 1989 (cf. Mao [5]).

Chapter II presents the explicit representation of the solutions to linear SIES. The results in Section 2.2 are classical and those in Section 2.3-2.5 are mainly based on my paper Mao [1] " Liouville's Formula for Stochastic Integral Equations " published in Journal of Fuzhou University in 1983 (cf. Mathematical Review 85g: 60069).

Chapter III contains the theory of stochastic stability and boundedness. Actually, we published three papers on the stability and boundedness of solutions to Itô's stochastic differential equations in Journal of East China Institute of Chemical Technology in 1983 and 1986 (cf. Li and Mao [1-3]). This chapter deals with the stability and boundedness of solutions to more general SIES. We first in Section 3.2 establish some new Lebesgue-Stieltjes inequalities of Gronwall-Bellman-Bihari type which will be used to prove the following results in Sections 3.3-3.6. We then establish systematically new results on stochastic stability and boundedness in Sections 3.3-3.6. The results in Sections 3.2 and 3.5 were published in Quarterly Journal of Mathematics Oxford (2), September 1989 (cf. Mao [4]).

Chapter IV is for comparison theorems. Section 4.2 deals with the comparison theorems between SIES and ordinary differential equations which are mainly based on our paper " On Comparison Theorems for a Kind of Integral Equations with Respect to Semimartingales " published in Journal of Engineering Mathematics in 1986. We next discuss the comparison theorems between two SIES in Section 4.3 via Lyapunov-like function that transforms a complicated SIES into a relatively simpler SIES.

Chapter V is devoted to the applications of the transformation formula which transforms stochastic integrals with respect to semimartingales into classical Itô's integrals with respect to Brownian motion. It should be pointed out that the transformation formula is nothing but the well-known technique of random time-change. The roots of this lie in the famous paper of Dubins and Schwartz [1], on time-changes of martingales, and for example the technique as applied to Brownian integrals is to be found in the celebrated monograph of McKean [1] and recent book of Karatzas and Shreve [1]. We apply this formula to study properties of SIES, for instance, the Markov property and stability. The results in Sections 5.2-5.3 were presented at the 17th International Conference on Stochastic Processes and Their Applications, 27 June - 1 July 1988, Roma, Italy. The results in Section 5.4 were published in the Proceedings of ICCON'89 IEEE International Conference on Control and Applications, April 1989 (cf. Mao [8]).

Finally, I would like to mention my some other research work during the two years of my Ph.D. study at Warwick. First of all, the inequalities in Section 3.2 are only a little part of my research work on Lebesgue-Stieltjes integral inequalities. On this field I have already completed several papers, for example, Mao [6, 11], that are going to be published in Chinese Journal of Mathematics. Secondly, I have also completed several papers on Random Fixed Point Theorems. Two of them, Mao [3, 10] were published in Stochastic Analysis and Applications in 1988 and 1989 respectively. Thirdly, I continue my research on SIES and publish several papers after finishing this thesis. For instance, I generalized the results in Section 3.5 and these generalized results, Mao [9], will be published in Quarterly Journal of Mathematics Oxford (2) in March 1990. I also developed the idea about the moment exponential stability discussed in Section 3.6 and studied the almost sure exponential stability. The results will appear in Stochastics and Stochastics Reports (cf. Mao [12]). In addition, the method developed in Section 5.4 was employed to investigate the exponential stability of another sort of delay SIES (different from that in Section 5.4) and the paper Mao [13] will appear in Stochastic Analysis and Applications.

Chapter I

Existence and Uniqueness of Solutions

1.1. INTRODUCTION

There are a lot of books on semimartingales, for example, Dellacherie and Meyer [1], Doob [1], Elliott [1], Elworthy [1], Jacod [1], Métivier [1], Rogers and Williams [1], Yan [1], hence we will use this knowledge without any explanation. In fact, we would have given a brief introduction to semimartingales if we had had an enough space.

In this chapter we will give several existence and uniqueness theorems of the solutions to SIES. Indeed, as early as in 1983 we had already published a paper in Stochastics which dealt with the existence and uniqueness of solutions to the following stochastic integral equation with respect to semimartingales (SIES)

$$X = \Phi(X) + F(X).M \tag{1.1.1}$$

where X was a one-dimensional stochastic process and M a semimartingale. It was Professor K. D. Elworthy who suggested that I study the following more general SIES

$$X = \Phi(X) + \sum_{i=1}^{m} F_i(X).M_i \tag{1.1.2}$$

where X was an n-dimensional stochastic process and M_i ($1 \le i \le m$) semimartingales, so that this thesis could be developed in more general case.

We first in section 1.2 introduce some necessary notations and basic assumptions. We next in section 1.3 give several fundamental existence and uniqueness theorems for Eq.(1.1.2) and then generalize these results in section 1.4. As an application we give several existence and uniqueness theorems of solutions to Doléans-Dade's equations in section 1.5. We also apply these results to study delay SIES in section 1.6 and delay Doléans-Dade's equations in section 1.7 respectively. the results in section 1.6 and 1.7 were published in Stochastic Analysis and Applications, March 1989 (cf. Mao [5]).

1.2. BASIC SETTING AND ASSUMPTIONS

Throughout this thesis we let (Ω, $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, P) be a complete probability sp with $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$.

Let $\mathcal{X}$ be the family of all n-dimensional cadlag. adapted processes. Denote by the set of all n-dimensional predictable processes. Let $\mathcal{O}$ represent the family of n-dimensional optional processes. In addition, let $\mathcal{M}$ stand for the set of all o dimensional semimartingales.

If $X \in \mathcal{X}$ and T is a stopping time, we define $X^{T-} = X \mathbb{1}_{[\![0,T[\![} + X(T-) \mathbb{1}_{[\![T,}$ and $X^T = X \mathbb{1}_{[\![0,T]\!]} + X(T) \mathbb{1}_{]\!]T,\infty[\![}$, where, and throughout this the $\mathbb{1}_B$ represents the indicator function of a set B. If $x = (x_1, ..., x_n) \in R^n$, we h $|x|^2 = \sum x_i^2$.

Let $1 \leq p \leq +\infty$. Denote $\mathcal{S}^p = \{X \in \mathcal{O}: \|X\|_{\mathcal{S}^p} < \infty\}$, where

$$\|X\|_{\mathcal{S}^p} = \|\sup_{t\geq 0} |X(t)|\|_{\mathcal{L}^p(\Omega)}$$

Let $M \in \mathcal{M}$ and $M = N + A$ a decomposition of M, where N is a local martingale a A is a process of finite variation. Define

$$j_p(N, A) = \| [N, N]_\infty^{1/2} + \int_{0-}^{\infty} |dA(s)| \|_{\mathcal{L}^p}$$

$$\|M\|_{\mathcal{H}^p} = \inf_{M=N+A} j_p(N, A)$$

where inf is over all the decompositions of M. Define $\mathcal{H}^p = \{M \in \mathcal{M} : \|M\| < \infty\}$. We also let $\mathcal{L}(M) = \{X \in \mathcal{P} : X \text{ is integrable with respect to } M\}$.

Let $M \in \mathcal{M}$, $1 \leq p \leq +\infty$, $a > 0$, $0 \leq \beta < 1$. Denote by $\mathcal{L}_M^p(a)$ the family of maps $F : \mathcal{X} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T, $F(X) \mathbb{1}_{]\!]0,T]\!]} = F(X^{T-}) \mathbb{1}_{]\!]0,T]\!]}$,

(2) for any $X, Y \in \mathcal{X}$, $\|F(X) - F(Y)\|_{\mathcal{S}^p} \leq a \|X - Y\|_{\mathcal{S}^p}$,

(3) $F(0) \in \mathcal{L}(M)$.

We also denote by $\mathcal{C}^p(\beta)$ the family of all maps $\Phi : \mathcal{X} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T, $\Phi(X) 1_{[\![0,T[\![} = \Phi(X \quad) 1_{[\![0,T[\![}$,

(ii) for any $X, Y \in \mathcal{X}$, $\|\Phi(X) - \Phi(Y)\|_{\mathcal{S}^p} \le \beta \|X - Y\|_{\mathcal{S}^p}$.

Yan [1] proved the following theorem.

Theorem 1.2.1 (Yan [1], Theorem 13.13). Let M_i $(1 \le i \le m) \in$ with $M_i(0) = 0$, $F_i \in \mathcal{L}^p_{M_i}(a)$, $\Phi \in \mathcal{C}^p(\beta)$, where $1 \le p < \infty$, $a > 0$, $0 \le \beta <$. Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.1.2).

Remark 1.2.2. In fact, Yan [1] only proved the existence and uni eness theorem for the one-dimensional Eq.(1.1.1). However, as his remark, s result can be extended to the above result without any difficulty.

In order to generalize this result we still need to introduce some new r ations. Let k be an integer and $X \in \mathcal{O}$. Define

$$X^{[k]} = X 1_{[0,k]}(|X|) + k(X/|X|) 1_{(k,\infty)}(|X|)$$

Let b be a positive constant and $M \in \mathcal{H}^\infty$. If there exist stopping t nes $\{T_j\}_{0 \le j \le k}$ with $0 = T_0 \le T_1 \le \dots \le T_k$ such that $M = M^{T_k-}$ and

$$\|(M - M^{T_{j-1}})^{T_j-}\|_{\mathcal{H}^\infty} \le b, \qquad j = 1, \dots, k$$

we say that M belongs to $\mathcal{D}(b)$ and $\{T_j\}_{0 \le j \le k}$ partition M into secti s with lengths less than b.

Let $1 \le p < \infty$, $M \in \mathcal{M}$, $\{a_k\}$ be a sequence of positive real number Denote by $L^p_M(\{a_k\})$ the set of all map $F : \mathcal{X} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F(X) 1_{]\!]0,T]\!]} = F(X^{T-}) 1_{]\!]0,T]\!]} \tag{1.2.1}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \dots$,

$$\|F(X^{[k]}) - F(Y^{[k]})\|_{\mathcal{S}^p} \le a_k \|X - Y\|_{\mathcal{S}^p} \tag{1.2.2}$$

(3) $F(0) \in \mathcal{L}(M)$.

Let $1 \le p < \infty$ and $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1$. Denote by

$C^p(\{\beta_k\})$ the family of all maps $\Phi : \mathcal{X} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Phi(X)\, 1_{[\![0, T [\![} = \Phi(X^{T-})\, 1_{[\![0, T [\![} \tag{1.2.3}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| \Phi(X^{[k]}) - \Phi(Y^{[k]}) \|_{\mathcal{S}^p} \leq \beta_k \| X - Y \|_{\mathcal{S}^p} \tag{1.2.4}$$

If $X \in \mathcal{O}$, define stopping time

$$T_k(X) = \inf\{ t \geq 0 : |X(t)| \geq k \} \tag{1.2.5}$$

for $k = 1, 2, \ldots,$ where, and throughout this thesis, we use $\inf \emptyset = \infty$.

We now can state our fundamental theorem.

1.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Theorem 1.3.1. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in L^p_{M_i}(\{a_k\})$, $\Phi \in C^p(\{\beta_k\})$, where $1 \leq p < \infty$, $\{a_k\}$ and $\{\beta_k\}$ are sequences of real numbers with $a_k > 0$ and $0 \leq \beta_k < 1$. Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k\to\infty} T_k(X) = \rho$ a.s. (which will be called the explosion time);

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{X}$ and

$$X^{\sigma-} = \Big(\Phi(X^{\sigma-}) + \sum_{i=1}^{m} F_i(X^{\sigma-}).M_i \Big)^{\sigma-} \tag{1.3.1}$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $Y^{\tau-} \in \mathcal{X}$ and

$$Y^{\tau-} = \Big(\Phi(Y^{\tau-}) + \sum_{i=1}^{m} F_i(Y^{\tau-}).M_i \Big)^{\tau-} \tag{1.3.2}$$

we have

(3) $\tau \leq \rho$ a.s. and

(4) $Y^{\tau-} = X^{\tau-}$ a.s.

In order to prove this theorem we need to prove a lemma first.

Lemma 1.3.2. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i, G_i \in \mathcal{L}^p_{M_i}(a)$, $\Phi, \Psi \in \mathcal{C}^p(\beta)$, where $1 \le p < \infty$, $a > 0$, $0 \le \beta < 1$. Let X and Y denote the unique solutions in $\mathcal{X}$ to the equations

$$X = \Phi(X) + \sum_{i=1}^{m} F_i(X).M_i$$

and

$$Y = \Psi(Y) + \sum_{i=1}^{m} G_i(Y).M_i$$

respectively. Let Q be a open set in R^n and define

$$T_Q(Z) = \inf\{ t \ge 0 : Z(t) \notin Q \}$$

for $Z = X, Y$. Clearly, $T_Q(Z)$ is a stopping time. If

$$\Phi(Z)^{T_Q(Z)-} = \Psi(Z)^{T_Q(Z)-}$$

$$F_i(Z)\, 1\!\!1_{]\!]0,T_Q(Z)]\!]} = G_i(Z)\, 1\!\!1_{]\!]0,T_Q(Z)]\!]}, \quad 1 \le i \le m$$

for $Z = X, Y$, then

$$T_Q(X) = T_Q(Y) \text{ a.s.} \quad \text{and} \quad X^{T_Q(X)-} = Y^{T_Q(Y)-} \text{ a.s.} \tag{1.3.3}$$

Proof. It follows from assumptions that

$$X^{T_Q(X)-} = \Phi(X)^{T_Q(X)-} + \sum_{i=1}^{m} F_i(X)\, 1\!\!1_{]\!]0,T_Q(X)]\!]} \cdot M_i^{T_Q(X)-}$$

$$= \Psi(X)^{T_Q(X)-} + \sum_{i=1}^{m} G_i(X)\, 1\!\!1_{]\!]0,T_Q(X)]\!]} \cdot M_i^{T_Q(X)-}$$

$$= \Psi(X^{T_Q(X)-})^{T_Q(X)-} + \sum_{i=1}^{m} G_i(X^{T_Q(X)-}) \,.\, M_i^{T_Q(X)-}$$

and

$$Y^{T_Q(X)-} = \Psi(Y^{T_Q(X)-})^{T_Q(X)-} + \sum_{i=1}^{m} G_i(Y^{T_Q(X)-}) \,.\, M_i^{T_Q(X)-}$$

Therefore, by Theorem 1.2.1, we obtain

$$X^{T_Q(X)-} = Y^{T_Q(X)-}$$

Hence, $T_Q(Y) \geq T_Q(X)$ a.s.. Similarly, we can also prove $T_Q(Y) \leq T_Q(X)$ a.s. and then we get desired result (1.3.3) immediately which completes the proof.

We now begin to prove Theorem 1.3.1.

Proof of Theorem 1.3.1. For every $i = 1, \ldots, m$ and $k = 1, 2, \ldots,$ define

$$\Phi_k(X) = \Phi(X^{[k]}), \quad F_{ik}(X) = F_i(X^{[k]}), \quad X \in \mathcal{X}$$

We can show $\Phi_k \in \mathcal{C}^p(\beta_k)$ and $F_{ik} \in \mathcal{L}^p_{M_i}(a_k)$ easily. Hence, by Theorem 1.2.1, there is a unique solution, say X_k, in $\mathcal{X}$ to the following equation

$$X = \Phi_k(X) + \sum_{i=1}^{m} F_{ik}(X).M_i \tag{1.3.4}$$

We now claim

$$T_k(X_k) = T_j(X_j) \quad \text{a.s.,} \quad 1 \leq k \leq j < \infty \tag{1.3.5}$$

$$X_k^{T_k(X_k)} = X_j^{T_j(X_j)} \quad \text{a.s.,} \quad 1 \leq k \leq j < \infty \tag{1.3.6}$$

In fact, for every $X \in \mathcal{X}$, we have

$$\begin{aligned}\Phi_k(X)^{T_k(X)-} &= \Phi(X^{[k]})^{T_k(X)-} = \Phi((X^{[k]})^{T_k(X)-})^{T_k(X)-} \\ &= \Phi(X^{T_k(X)-})^{T_k(X)-} = \Phi_j(X)^{T_k(X)-}\end{aligned} \tag{1.3.7}$$

and

$$\begin{aligned}&F_{ik}(X)\, 1\!\!1_{]\!]0, T_k(X)]\!]} = F((X^{[k]})^{T_k(X)-})\, 1\!\!1_{]\!]0, T_k(X)]\!]} \\ &= F(X^{T_k(X)-})\, 1\!\!1_{]\!]0, T_k(X)]\!]} = F_{ij}(X)\, 1\!\!1_{]\!]0, T_k(X)]\!]}\end{aligned} \tag{1.3.8}$$

Using Lemma 1.3.2, we deduce (1.3.5) and (1.3.6) immediately which imply that $T_k(X_k) = T_k(X_{k+1}) \leq T_{k+1}(X_{k+1})$, i.e., $T_k(X_k) \uparrow$ as $k \uparrow$. Define

$$\rho = \lim_{k \to \infty} T_k(X_k)$$

$$X = X_1\, 1\!\!1_{[\![0,\, T_1(X_1)]\!]} + \sum_{k=2}^{\infty} X_k\, 1\!\!1_{]\!]T_{k-1}(X_{k-1}),\, T_k(X_k)]\!]}$$

It is obvious that ρ is a stopping time, $X \in \mathcal{O}$ and

$$T_k(X) = T_k(X_k) \quad \text{and} \quad X^{T_k(X)-} = X_k^{T_k(X_k)-} \quad \text{a.s.} \qquad (1.3.9)$$

for all $k = 1, 2, \ldots$. Hence we have $\rho = \lim_{k\to\infty} T_k(X)$ a.s. which is desired result (1).

Let σ is a stopping time with $\sigma < \rho$ a.s. Clearly, $X^{\sigma-} \in \mathcal{X}$. We also have

$$X^{T_k(X)\wedge\sigma-} = \Phi_k(X_k)^{T_k(X_k)\wedge\sigma-} + \sum_{i=1}^{m} F_{ik}(X_k).M_i^{T_k(X_k)\wedge\sigma-}$$

$$= \Phi_k((X_k)^{T_k(X_k)\wedge\sigma-})^{T_k(X_k)\wedge\sigma-} + \sum_{i=1}^{m} F_{ik}((X_k)^{T_k(X_k)\wedge\sigma-}).M_i^{T_k(X_k)\wedge\sigma-}$$

$$= \Phi_k(X^{T_k(X)\wedge\sigma-})^{T_k(X)\wedge\sigma-} + \sum_{i=1}^{m} F_{ik}(X^{T_k(X)\wedge\sigma-}).M_i^{T_k(X_k)\wedge\sigma-}$$

$$= \Phi(X^{T_k(X)\wedge\sigma-})^{T_k(X)\wedge\sigma-} + \sum_{i=1}^{m} F_i(X^{T_k(X)\wedge\sigma-}).M_i^{T_k(X_k)\wedge\sigma-}$$

$$= (\Phi(X^{\sigma-}) + \sum_{i=1}^{m} F_i(X^{\sigma-}).M_i)^{T_k(X)\wedge\sigma-} \qquad (1.3.10)$$

which deduces required result (1.3.1) immediately. We now assume $Y \in \mathcal{O}$ and τ is a stopping time such that $Y^{\tau-} \in \mathcal{X}$ and Eq.(1.3.2) holds. In the same way as the proof of Lemma 1.3.2 we can prove

$$T_k(Y)\wedge\tau \le T_k(X_k)\wedge\tau \quad \text{and} \quad Y^{T_k(Y)\wedge\tau-} = X_k^{T_k(Y)\wedge\tau-} \quad \text{a.s.} \qquad (1.3.11)$$

We now show $\tau \le \rho$ a.s. If it not true, then for almost all $\omega \in \{\omega : \tau > \rho\}$ we have from (1.3.11), (1.3.9) and conclusion (1) that

$$T_k(Y)\wedge\tau(\omega) \le \lim_{k\to\infty} T_k(X)\wedge\tau(\omega) = \rho\wedge\tau(\omega) = \rho(\omega)$$

i.e., $T_k(Y)(\omega) \le \rho(\omega) < \infty$ for all $k = 1, 2, \ldots$. Hence, we get

$$\lim_{k\to\infty} Y(T_k(Y)(\omega), \omega) = \lim_{k\to\infty} k = \infty$$

which is a contradiction since $Y^{\tau-} \in \mathcal{X}$. Finally, we can similarly prove

$$\lim_{k\to\infty} T_k(Y) \ge \tau \quad \text{a.s.}$$

which, together with (1.3.9) and (1.3.11), implies desired assertion (4) and we complete the proof.

Comparing Theorem 1.3.1 with 1.2.1, we can find that the explosion time $\rho = \infty$ a.s. if Φ and F_i are all Lipschitz continuous. In the sequel we will give another condition, i.e., the linear growth condition, to ensure $\rho = \infty$ a.s.

Let $1 \le p < \infty$, $K_1 > 0$ and $\{a_k\}$ be a sequence of positive real numbers. Denote by $\mathcal{L}^p(\{a_k\}, K_1)$ the set of all maps $F : \mathcal{X} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F(X)\,1_{]\!]0,T]\!]} = F(X^{T-})\,1_{]\!]0,T]\!]} \tag{1.3.12}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\|(F(X^{[k]}) - F(Y^{[k]}))\|_{\varphi^p} \le a_k \|X - Y\|_{\varphi^p} \tag{1.3.13}$$

(3) for any $X \in \mathcal{X}$,

$$\|F(X)\|_{\varphi^p} \le K_1(1 + \|X\|_{\varphi^p}) \tag{1.3.14}$$

Let $1 \le p < \infty$, $K_2 \ge 0$, $0 \le \alpha < 1$, $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1$. Denote by $\mathcal{C}^p(\{\beta_k\}, \alpha, K_2)$ the family of all maps $\Phi : \mathcal{X} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Phi(X)\,1_{[\![0,T[\![} = \Phi(X^{T-})\,1_{[\![0,T[\![} \tag{1.3.15}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\|\Phi(X^{[k]}) - \Phi(Y^{[k]})\|_{\varphi^p} \le \beta_k \|X - Y\|_{\varphi^p} \tag{1.3.16}$$

(iii) for any $X \in \mathcal{X}$,

$$\|\Phi(X)\|_{\varphi^p} \le K_2 + \alpha \|X\|_{\varphi^p} \tag{1.3.17}$$

Theorem 1.3.3. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in \mathcal{L}^p(\{a_k\}, K_1)$, $\Phi \in \mathcal{C}^p(\{\beta_k\}, \alpha, K_2)$, where $1 \le p < \infty$, $K_1 > 0$, $K_2 \ge 0$, $0 \le \alpha < 1$, $\{a_k\}$ and $\{\beta_k\}$ are sequences of real numbers with $a_k > 0$ and $0 \le \beta_k < 1$. Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.1.2).

In order to prove this theorem we need some lemmas. We first state some lemmas due to Emery [1].

Lemma 1.3.4 (Emery [1]). Let $M \in \mathcal{M}$. Then

1) For any stopping time T we have

$$\|M^T\|_{\mathcal{H}^\infty} \leq \|M\|_{\mathcal{H}^\infty}, \qquad \|M^{T-}\|_{\mathcal{H}^\infty} \leq 2\|M\|_{\mathcal{H}^\infty}$$

2) For arbitrary $b > 0$, there exists a sequence of stopping times $\{T_k\}$ with $T_0 = 0$ and $T_k \uparrow \infty$ such that $T_k > T_{k-1}$ on $\{\omega: T_{k-1} < \infty\}$ and

$$\|M^{T_k-} - M^{T_{k-1}}\|_{\mathcal{H}^\infty} \leq b$$

Lemma 1.3.5 (Emery [1]). (1) If $M \in \mathcal{D}(b)$ ($b > 0$), then $M^{T-} \in \mathcal{D}(2b)$ for any stopping time T.

(2) Let $M \in \mathcal{M}$. Then for arbitrary $b > 0$, there exist stopping times $T_k \uparrow \infty$ a.s. such that $M^{T_k-} \in \mathcal{D}(b)$ for all $k = 1, 2, \cdots$.

Lemma 1.3.6 (Emery [1]). (1) Let $1 \leq p < \infty$. Then there exists a constant $C_p > 1$ such that for all $M \in \mathcal{H}^p$ we have

$$\|M\|_{\mathcal{S}^p} \leq C_p \|M\|_{\mathcal{H}^p}$$

(2) Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $1 \leq r \leq \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$. If $M \in \mathcal{H}^p$ and $X \in \mathcal{P} \cap \mathcal{S}^p$, then X is integrable with respect to M and

$$\|X.M\|_{\mathcal{H}^r} \leq \|X\|_{\mathcal{S}^p} \|M\|_{\mathcal{H}^q}$$

We now prove some new lemmas.

Lemma 1.3.7. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in \mathcal{L}^p_{M_i}(a)$, $\Phi \in \mathcal{C}^p(\beta)$, where $1 \leq p < \infty$, $a > 0$, $0 \leq \beta < 1$. Let X denote the unique solution in $\mathcal{X}$ to the equation

$$X = \Phi(X) + \sum_{i=1}^{m} F_i(X).M_i$$

Let $0 \leq \alpha < 1$, $K_1 > 0$, $K_2 > 0$, $0 < b < (1-\alpha)/(m\, C_p\, K_1)$. If

(1) for all $X \in \mathcal{X}$,

$$\|F(X)\|_{\mathcal{S}^p} \leq K_1(1 + \|X\|_{\mathcal{S}^p}) \tag{1.3.18}$$

$$\|\Phi(X)\|_{\mathcal{S}^p} \leq K_2 + \alpha\|X\|_{\mathcal{S}^p} \tag{1.3.19}$$

(2) $M \in \mathcal{H}^\infty$ and there exists a sequence of stopping times $\{T_k\}_{k\geq 0}$ with $T_0 = 0$ and $T_k \uparrow \infty$ such that $T_k > T_{k-1}$ on $\{\omega: T_{k-1} < \infty\}$ and

$$\| M_i^{T_k-} - M_i^{T_{k-1}} \|_{\mathcal{X}^\infty} \leq b \quad \text{for all } 1 \leq i \leq m \text{ and } k = 1, 2, \ldots$$

Then $X^{T_k-} \in \mathcal{G}^p$ and

$$\| X^{T_k-} \|_{\mathcal{G}^p} \leq \frac{1}{1 - \alpha - m\, C_p\, b\, K_1} \cdot \frac{\delta^k - 1}{\delta - 1}$$

$$\times \left(K_2 + m\, C_p\, b\, K_1 + 2\, C_p\, K_1 \sum_{i=1}^{m} \| M_i \|_{\mathcal{X}^\infty} \right) \tag{1.3.20}$$

where

$$\delta = \frac{1}{1 - \alpha - m\, C_p\, b\, K_1}\, 2\, C_p\, K_1 \sum_{i=1}^{m} \| M_i \|_{\mathcal{X}^\infty}$$

and $(\delta^k - 1)/(\delta - 1)$ will be defined to be k if $\delta = 1$.

Proof. For $0 < l \leq k$, we have

$$X^{T_l-} = \Phi(X)^{T_l-} + \sum_{i=1}^{m} F_i(X).M_i^{T_l-}$$

$$= \Phi(X^{T_l-})^{T_l-} + \sum_{i=1}^{m} F_i(X^{T_l-}).(M_i - M_i^{T_{l-1}})^{T_l-} - \sum_{i=1}^{m} F_i(X^{T_{l-1}-}).(M_i^{T_{l-1}})^{T_l-}$$

Since, by Lemma 1.3.4 and 1.3.6 and assumption (1.3.18), we have inequalities

$$\| F_i(X^{T_l-}).(M_i - M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{G}^p}$$

$$\leq C_p \| F_i(X^{T_l-}).(M_i - M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{X}^p}$$

$$\leq C_p \| F_i(X^{T_l-}) \|_{\mathcal{G}^p} \| (M_i - M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{X}^\infty}$$

$$\leq C_p\, b\, K_1 (1 + \| X^{T_l-} \|_{\mathcal{G}^p})$$

and

$$\| F_i(X^{T_{l-1}-}).(M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{G}^p}$$

$$\leq C_p \| F_i(X^{T_{l-1}-}).(M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{X}^p}$$

$$\leq C_p \, \| F_i(X^{T_{l-1}-}) \|_{\varphi^p} \, \| (M_i^{T_{l-1}})^{T_l-} \|_{\mathcal{H}^\infty}$$

$$\leq 2 C_p K_1 \, \| M_i \|_{\mathcal{H}^\infty} (1 + \| X^{T_{l-1}-} \|_{\varphi^p})$$

we get

$$\| X^{T_l-} \|_{\varphi^p} \leq K_2 + \alpha \| X^{T_l-} \|_{\varphi^p} + m C_p b K_1 (1 + \| X^{T_l-} \|_{\varphi^p})$$

$$+ 2 C_p K_1 (1 + \| X^{T_{l-1}-} \|_{\varphi^p}) \sum_{i=1}^{m} \| M_i \|_{\mathcal{H}^\infty}$$

By induction, we can easily prove $\| X^{T_l-} \|_{\varphi^p} < \infty$. Thus

$$\| X^{T_l-} \|_{\varphi^p} \leq \frac{1}{1 - \alpha - m C_p b K_1} \Big(K_2 + m C_p b K_1 + 2 C_p b K_1 \sum_{i=1}^{m} \| M_i \|_{\mathcal{H}^\infty} \Big)$$

$$+ \delta \, \| X^{T_{l-1}-} \|_{\varphi^p}$$

Therefore, by induction again, we deduce desired result (1.3.20) and complete the proof.

Lemma 1.3.8. Let M_i $(1 \leq i \leq m) \in \mathcal{H}^\infty$ with $M_i(0) = 0$. Then for arbitrary $b > 0$, there exist a number of stopping times $\{ T_k \}_{0 \leq k \leq l}$ with $0 = T_0 \leq T_1 \leq \dots \leq T_l = \infty$ a.s. such that $T_k > T_{k-1}$ on $\{ T_{k-1} < \infty \}$ and

$$\| M_i^{T_k-} - M_i^{T_{k-1}} \|_{\mathcal{H}^\infty} \leq b \quad \text{for all } 1 \leq i \leq m \text{ and } 1 \leq k \leq l \tag{1.3.21}$$

Proof. Since $M_i \in \mathcal{H}^\infty$, there exists a decomposition $M_i = N_i + A_i$ of M_i with N_i a local martingale and A_i a process of finite variation such that $j_\infty(N_i, A_i) < \infty$. Define stopping times

$$T_0 = 0$$

$$T_k = \inf \{ t > T_{k-1} : \text{either } [N_i, N_i]_t - [N_i, N_i]_{T_{k-1}} > b^2/4$$

$$\text{or} \int_{T_{k-1}}^{t} | dA_i(s) | > b/2 \quad \text{for some } i = 1, 2, \dots, m \}, \quad k \geq 1$$

It is clear that $T_k > T_{k-1}$ on $\{ T_{k-1} < \infty \}$ for all $k \geq 1$. We then claim that for sufficiently large k, we have $T_k = \infty$ a.s. Otherwise, we have

$$P(T_k < \infty) > 0 \text{ for all } k = 1, 2, \dots$$

In this case, let $\omega \in (T_k < \infty)$ and we then have

$$\sum_{i=1}^{m}\Big([N_i, N_i]_{\infty}^{1/2} + \int_{0-}^{\infty} |dA_i(s)|\Big)(\omega)$$

$$\geq \sum_{i=1}^{m}\Big([N_i, N_i]_{T_k}^{1/2} + \int_{0-}^{T_k} |dA_i(s)|\Big)(\omega)$$

$$\geq \sum_{i=1}^{m}\Big(\big(\sum_{j=1}^{k}([N_i, N_i]_{T_j} - [N_i, N_i]_{T_{j-1}})\big)^{1/2} + \sum_{j=1}^{k}\int_{T_{j-1}}^{T_j} |dA_i(s)|\Big)(\omega)$$

$$\geq \Big(\sum_{i=1}^{m}\sum_{j=1}^{k}\big([N_i, N_i]_{T_j} - [N_i, N_i]_{T_{j-1}} + \big(\int_{T_{j-1}}^{T_j} |dA_i(s)|\big)^2\big)\Big)^{1/2}(\omega)$$

$$\geq \Big(\sum_{j=1}^{k}\max_{1\leq i\leq m}\big(([N_i, N_i]_{T_j} - [N_i, N_i]_{T_{j-1}}) \vee \big(\int_{T_{j-1}}^{T_j} |dA_i(s)|\big)^2\big)\Big)^{1/2}(\omega)$$

$$\geq (k\, b^2/4)^{1/2} = \frac{b}{2}\sqrt{k}$$

which deduce

$$\infty > \sum_{i=1}^{m} j_{\infty}(N_i, A_i) = \sum_{i=1}^{m}\Big\| [N_i, N_i]_{\infty}^{1/2} + \int_{0-}^{\infty} |dA_i(s)| \Big\|_{\mathcal{L}^{\infty}} \geq \frac{b}{2}\sqrt{k}$$

for all $k = 1, 2, \ldots$. This is a contradiction.

We now let l be large enough so that $T_l = \infty$ a.s. For all $1 \leq i \leq m$ and $1 \leq k \leq l$, we then have

$$M_i^{T_k-} - M_i^{T_{k-1}} = (N_i^{T_k-} - N_i^{T_{k-1}}) + (A_i^{T_k-} - A_i^{T_{k-1}}) := N_{ik} + A_{ik}$$

However

$$[N_{ik}, N_{ik}]_{\infty} = [N_i, N_i]_{T_k-} - [N_i, N_i]_{T_{k-1}} \leq b^2/4$$

and

$$\int_{0-}^{\infty} |dA_{ik}(s)| = \int_{T_{k-1}}^{T_k-} |dA_i(s)| \leq b/2$$

Therefore

$$\| M_i^{T_k-} - M_i^{T_{k-1}} \|_{\mathcal{X}^{\infty}} \leq j_{\infty}(N_{ik}, A_{ik}) \leq b$$

The proof has been completed.

We now can prove Theorem 1.3.3.

Proof of Theorem 1.3.3. We use the same notations as the proof of Theorem 1.3.1. By Theorem 1.3.1, for the existence we only need to prove $\rho = \infty$. Set $0 < b < (1 - \alpha)/(m\, C_p\, K_1)$. We first show that there are stopping times $U_k \uparrow \infty$ a.s. such that $M_i^{U_k-} \in \mathcal{H}^\infty$ for all $1 \le i \le m$ and $k \ge 1$, and, in particular, we can take stopping times $0 = U_{k0} \le U_{k1} \le \dots \le U_{kk} \ge U_k$ a.s. such that $\| M_i^{U_{kj}-} - M_i^{U_{k(j-1)}} \|_{\mathcal{H}^\infty} \le b$ and $U_{kj} > U_{k(j-1)}$ on $\{ U_{k(j-1)} < \infty \}$. Indeed, by Lemma 1.3.5, there exist stopping times $V_k \uparrow \infty$ a.s. such that $M_i^{V_k-} \in \mathcal{H}^\infty$ for all $1 \le i \le m$ and $k \ge 1$. By Lemma 1.3.8, there exist a unmber of stopping times $0 = V_{11} \le V_{12} \le \dots \le V_{1k_1} \ge V_1$ a.s. such that $\| M_i^{V_{1j}-} - M_i^{V_{1(j-1)}} \|_{\mathcal{H}^\infty} \le b$ and $V_{1j} > V_{1(j-1)}$ on $\{ V_{1(j-1)} < \infty \}$. By induction, suppose we have determined k_u. By Lemma 1.3.8 again, there exist a unmber of stopping times $0 = V_{k_u 1} \le V_{k_u 2} \le \dots \le V_{k_u k_{u+1}} \ge V_{k_u}$ a.s. such that $\| M_i^{V_{k_u j}-} - M_i^{V_{k_u (j-1)}} \|_{\mathcal{H}^\infty} \le b$ and $V_{k_u j} > V_{k_u (j-1)}$ on $\{ V_{k_u (j-1)} < \infty \}$. Particularly, we can choose $k_{u+1} > k_u$. We now define

$$U_k = \begin{cases} 0, & \text{if } 1 \le k < k_1 \\ V_{k_u}, & \text{if } k_{u+1} \le k < k_{u+2} \end{cases}$$

for $k = 1, 2, \dots$, which satisfy our requirments above. Therefore, using Lemma 1.3.7, we have

$$\| X_j^{U_k-} \|_{\mathcal{S}^p} \le \frac{1}{1 - \alpha - m\, C_p\, b\, K_1} \cdot \frac{(\delta_k)^k - 1}{\delta_k - 1} \times$$

$$\times \Big(K_2 + m\, C_p\, b\, K_1 + 2\, C_p\, K_1 \sum_{i=1}^m \| M_i^{U_k-} \|_{\mathcal{H}^\infty} \Big) \tag{1.3.22}$$

for all $k, j = 1, 2, \dots$, where

$$\delta_k = \frac{1}{1-\alpha-m\,C_p\,b\,K_1}\,2\,C_p\,K_1\sum_{i=1}^{m}\|M_i^{U_k-}\|_{\mathcal{H}^\infty}$$

and $\dfrac{(\delta_k)^k-1}{\delta_k-1}$ will be defined to be k if $\delta_k = 1$. We claim that

$$\underline{\lim}_{j\to\infty} T_j(X_j) \geq U_k \quad \text{a.s. for all } k = 1, 2, \ldots \tag{1.3.23}$$

If it is not true, set $\Omega_k = \{\omega : \underline{\lim}_{j\to\infty} T_j(X_j) < U_k\}$ and then $P(\Omega_k) > 0$ for some k.

Hence for almost all $\omega \in \Omega_k$, we have $T_j(X_j) < U_k$, i.e., $\sup_{t\geq 0} |X_j^{U_k-}(t)| \geq j$. Thus

$$\|X_j^{U_k-}\|_{\mathcal{S}^p} \geq j\,P(\Omega_k) \qquad \text{for all } j = 1, 2, \ldots$$

which is in contradiction with (1.3.22). Therefore, using(1.3.23) and (1.3.9) we get $\rho = \infty$ a.s. and complete the proof of existence.

We now begin to prove the uniqueness. Let X and Y be two solutions of Eq.(1.3.1) in $\mathcal{X}$. Set $Q_k = T_k(X) \wedge T_k(Y)$ for $k = 1, 2, \ldots$ and we have $Q_k \uparrow \infty$ a.s. clearly. We also have

$$X^{Q_k-} = \Phi(X^{Q_k-})^{Q_k-} + \sum_{i=1}^{m} F_i(X^{Q_k-}).M_i^{Q_k-}$$

$$= \Phi_k(X^{Q_k-})^{Q_k-} + \sum_{i=1}^{m} F_{ik}(X^{Q_k-}).M_i^{Q_k-}$$

and

$$Y^{Q_k-} = \Phi_k(Y^{Q_k-})^{Q_k-} + \sum_{i=1}^{m} F_{ik}(Y^{Q_k-}).M_i^{Q_k-}$$

It is easy to check that $\Phi_k(.)^{Q_k-} \in \mathcal{C}^p(\beta_k)$ and $F_{ik}(.) \in \mathcal{L}^p_{M_i^{Q_k-}}(a_k)$. Hence an application of Theorem 1.2.1 implies

$$X^{Q_k-} = Y^{Q_k-} \quad \text{for all } k = 1, 2, \ldots$$

which means X = Y. The proof of uniqueness is also complete.

From the proof we have the following estimate of the solutions immediately.

Corollary 1.3.9. Under the assumptions of Theorem 1.3.3, for arbitrary $0 < b < (1-\alpha)/(m\,C_p\,K_1)$, there exists a sequence of stopping times $\{U_k\}$ with $U_k \uparrow \infty$ a.s. such that $M_i^{U_k-} \in \mathcal{H}^\infty$ for all $1 \le i \le m$ and $k \ge 1$ and the unique solution X of Eq.(1.1.2) in $\mathcal{X}$ satisfies the estimate

$$\|X^{U_k-}\|_{\mathcal{S}^p} \le \frac{1}{1-\alpha-m\,C_p\,b\,K_1}\cdot\frac{(\delta_k)^k-1}{\delta_k-1}$$

$$\cdot\Big(K_2 + m\,C_p\,b\,K_1 + 2\,C_p\,K_1\sum_{i=1}^{m}\|M_i^{U_k-}\|_{\mathcal{H}^\infty}\Big) \tag{1.3.24}$$

for all $k = 1, 2, \ldots,$ where

$$\delta_k = \frac{1}{1-\alpha-m\,C_p\,b\,K_1}\,2\,C_p\,K_1\sum_{i=1}^{m}\|M_i^{U_k-}\|_{\mathcal{H}^\infty}$$

and $\dfrac{(\delta_k)^k-1}{\delta_k-1}$ will be defined to be k if $\delta_k = 1$.

Proof. We have

$$\|(X^{U_k-})^{T_j(X)-}\|_{\mathcal{S}^p} \le \|(X^{T_j(X)-})^{U_k-}\|_{\mathcal{S}^p}$$

$$\le \|((X_j)^{T_j(X_j)-})^{U_k-}\|_{\mathcal{S}^p} \le \|X_j^{U_k-}\|_{\mathcal{S}^p}$$

which, together with (1.3.22) and Fatou's convergence Lemma, yields required (1.3.24). The proof is complete.

1.4. GENERALIZATIONS

In this section we first generalize Theorem 1.3.1 and 1.3.3 and then we give several existence and uniqueness theorems of semimartingale solutions to Eq.(1.1.2).

Let $1 \leq p < \infty$, $M \in \mathcal{M}$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ nondecreasing sequence of stopping times. Denote by $L^p_M(\{a_k\}, \{S_k\})$ the set of a maps $F : \mathcal{X} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F(X)\, 1\!\!1_{]\!]0,T]\!]} = F(X^{T-})\, 1\!\!1_{]\!]0,T]\!]} \tag{1.4.1}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\|(F(X^{[k]}) - F(Y^{[k]}))\, 1\!\!1_{]\!]0,S_k]\!]}\|_{\varphi^p} \leq a_k \|X - Y\|_{\varphi^p} \tag{1.4.2}$$

(3) $F(0)\, 1\!\!1_{]\!]0,S_k]\!]} \in \mathcal{L}(M)$ for all $k = 1, 2, \ldots$.

Let $1 \leq p < \infty$, $\{\beta_k\}$ be a sequence of real numbers with $0 \leq \beta_k < 1$ and $\{S_k\}$ nondecreasing sequence of stopping times. Denote by $\mathcal{C}^p(\{\beta_k\}, \{S_k\})$ the family c all maps $\Phi : \mathcal{X} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Phi(X)\, 1\!\!1_{[\![0,T[\![} = \Phi(X^{T-})\, 1\!\!1_{[\![0,T[\![} \tag{1.4.3}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\|(\Phi(X^{[k]}) - \Phi(Y^{[k]}))^{S_k-}\|_{\varphi^p} \leq \beta_k \|X - Y\|_{\varphi^p} \tag{1.4.4}$$

The following theorem is a generalization of Theorem 1.3.1 and can be proved in th same way as Theorem 1.3.1.

Theorem 1.4.1. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in L^p_{M_i}(\{a_k\}, \{S_k\})$, $\Phi \in \mathcal{C}^p(\{\beta_k\}, \{S_k\})$, where $1 \leq p < \infty$, $\{a_k\}$ and $\{\beta_k\}$ are sequences of real numbers with $a_k > 0$ and $0 \leq \beta_k < 1$ and $\{S_k\}$ a sequence of nondecreasing stopping times. Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k\to\infty} T_k(X) \wedge S_k = \rho$ a.s.;

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{X}$ and

$$X^{\sigma-} = \left(\Phi(X^{\sigma-}) + \sum_{i=1}^{m} F_i(X^{\sigma-}).M_i\right)^{\sigma-} \tag{1.4.5}$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $X^{\tau-} \in \mathcal{X}$ and

$$Y^{\tau-} = \left(\Phi(Y^{\tau-}) + \sum_{i=1}^{m} F_i(Y^{\tau-}).M_i \right)^{\tau-} \tag{1.4.6}$$

we have

(3) $\rho' := \lim_{k\to\infty} S_k \wedge \tau \leq \rho$ a.s. and $Y^{\rho'-} = X^{\rho'-}$;

(4) $\tau \leq \rho$ a.s. on $\left\{ \omega : \lim_{k\to\infty} T_k(X) \leq \lim_{k\to\infty} S_k \right\}$.

Let $1 \leq p < \infty$, $\{ a_k \}$ and $\{ K_{1k} \}$ be two sequences of positive real numbers, $\{ S_k \}$ be a sequence of stopping times such that $S_k \uparrow \infty$ a.s. Denote by $\underline{L}^p(\{a_k\}, \{K_{1k}\}, \{S_k\})$ the family of all maps $F : \mathcal{X} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F(X)\, \mathbb{1}_{]\!]0,T]\!]} = F(X^{T-})\, \mathbb{1}_{]\!]0,T]\!]} \tag{1.4.7}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots$,

$$\| (F(X^{[k]}) - F(Y^{[k]}))\, \mathbb{1}_{]\!]0,S_k]\!]} \|_{\varphi^p} \leq a_k \| X - Y \|_{\varphi^p} \tag{1.4.8}$$

(3) for any $X \in \mathcal{X}$ and $k = 1, 2, \ldots$,

$$\| F(X)\, \mathbb{1}_{]\!]0,S_k]\!]} \|_{\varphi^p} \leq K_{1k}(1 + \| X \|_{\varphi^p}) \tag{1.4.9}$$

Let $1 \leq p < \infty$. Let $\{\beta_k\}$, $\{\alpha_k\}$, $\{K_{2k}\}$ be three sequences of nonnegative real numbers with $0 \leq \beta_k < 1$ and $0 \leq \alpha_k < 1$. Let $\{ S_k \}$ be a sequence of stopping times such that $S_k \uparrow \infty$ a.s. Denote by $\underline{C}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$ the family of all maps $\Phi : \mathcal{X} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Phi(X)\, \mathbb{1}_{[\![0,T[\![} = \Phi(X^{T-})\, \mathbb{1}_{[\![0,T[\![} \tag{1.4.10}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots$,

$$\| (\Phi(X^{[k]}) - \Phi(Y^{[k]}))^{S_k-} \|_{\varphi^p} \leq \beta_k \| X - Y \|_{\varphi^p} \tag{1.4.11}$$

(iii) for any $X \in \mathcal{X}$ and $k = 1, 2, \ldots$,

$$\| (\Phi(X))^{S_k-} \|_{\varphi^p} \leq K_{2k} + \alpha_k \| X \|_{\varphi^p} \tag{1.4.12}$$

The following theorem is a generalization of Theorem 1.3.3.

Theorem 1.4.2. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in \underline{L}^p(\{a_k\}, \{K_{1k}\},$

$\{S_k\}$), $\Phi \in \underline{C}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$, where $1 \leq p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ are two sequences of positive real numbers, $\{\beta_k\}$, $\{\alpha_k\}$, $\{K_{2k}\}$ are three sequences of nonnegative real numbers with $0 \leq \beta_k < 1$ and $0 \leq \alpha_k < 1$, and $\{S_k\}$ is a sequence of stopping times such that $S_k \uparrow \infty$ a.s.. Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.1.2).

Proof. It is easy to check that $\Phi(\cdot)^{S_k-} \in \mathcal{C}^p(\{\beta_l\}, \alpha_k, K_{2k})$ and $F_i(\cdot)\, 1\!\!1_{]\!]0,\, S_k]\!]} \in \mathcal{L}^p(\{a_l\}, K_{1k})$. Hence, by Theorem 1.3.3, there is a unique solution X^k in $\mathcal{X}$ to the equation

$$X = \Phi(X)^{S_k-} + \sum_{i=1}^{m} F_i(X)\, 1\!\!1_{]\!]0,\, S_k]\!]} \cdot M_i^{S_k-} \tag{1.4.13}$$

We can easily verify

$$(X^{k+1})^{S_k-} = (X^k)^{S_k-}, \qquad k = 1, 2, \dots$$

Therefore, there exists a unique X in $\mathcal{X}$ such that

$$X^{S_k-} = (X^k)^{S_k-}, \qquad k = 1, 2, \dots$$

Consequently,

$$\begin{aligned} X^{S_k-} &= \Phi(X^{S_k-})^{S_k-} + \sum_{i=1}^{m} F_i(X^{S_k-})\, 1\!\!1_{]\!]0,\, S_k]\!]} \cdot M_i^{S_k-} \\ &= \Phi(X)^{S_k-} + \sum_{i=1}^{m} F_i(X)\, 1\!\!1_{]\!]0,\, S_k]\!]} \cdot M_i^{S_k-} \\ &= \Phi(X)^{S_k-} + \sum_{i=1}^{m} F_i(X) \,.\, M_i^{S_k-} \\ &= \Big(\Phi(X) + \sum_{i=1}^{m} F_i(X) \,.\, M_i \Big)^{S_k-} \end{aligned}$$

which means X is a solution of Eq.(1.1.2). Uniqueness is obvious. We are done.

In order to give the existence and uniqueness theorems of semimartingale solutions to Eq.(1.1.2), we denote by $\mathcal{M}_n$ the set of of n-dimensional semimartingales.

Let $1 \leq p < \infty$, $M \in \mathcal{M}$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ a nondecreasing sequence of stopping times. Denote by $\mathcal{M}_M^p(\{a_k\}, \{S_k\})$ the set of all maps $F : \mathcal{M}_n \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$F(X)\, 1_{]\!]0,T]\!]} = F(X^{T-})\, 1_{]\!]0,T]\!]} \tag{1.4.14}$$

(2) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots$,

$$\| (F(X^{T_k(X)-}) - F(Y^{T_k(X)-}))\, 1_{]\!]0,S_k]\!]} \|_{\mathcal{P}^p} \le a_k \| X - Y \|_{\mathcal{H}^p} \tag{1.4.15}$$

(3) $F(0)\, 1_{]\!]0,S_k]\!]} \in \mathcal{L}(M)$ for all $k = 1, 2, \ldots$.

Let $1 \le p < \infty$, $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1$ and $\{S_k\}$ a nondecreasing sequence of stopping times. Denote by $\mathcal{N}^p(\{\beta_k\}, \{S_k\})$ the family of all maps $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ such that

(i) for any $X \in \mathcal{M}_n$ and stopping time T,

$$\Phi(X)\, 1_{[\![0,T[\![} = \Phi(X^{T-})\, 1_{[\![0,T[\![} \tag{1.4.16}$$

(ii) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots$,

$$\| (\Phi(X^{T_k(X)-}) - \Phi(Y^{T_k(X)-}))^{S_k-} \|_{\mathcal{P}^p} \le \beta_k \| X - Y \|_{\mathcal{H}^p} \tag{1.4.17}$$

In the samw way as the proof of Theorem 1.4.1 we can prove the following theorem.

Theorem 1.4.3. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in \mathcal{M}^p_{M_i}(\{a_k\}, \{S_k\})$, $\Phi \in \mathcal{N}^p(\{\beta_k\}, \{S_k\})$, where $1 \le p < \infty$, $\{a_k\}$ and $\{\beta_k\}$ are sequences of real numbers with $a_k > 0$ and $0 \le \beta_k < 1$ and $\{S_k\}$ a sequence of nondecreasing stopping times. Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k\to\infty} T_k(X) \wedge S_k = \rho$ a.s.;

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{M}_n$ and

$$X^{\sigma-} = \left(\Phi(X^{\sigma-}) + \sum_{i=1}^{m} F_i(X^{\sigma-}).M_i \right)^{\sigma-} \tag{1.4.18}$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $X^{\tau-} \in \mathcal{M}_n$ and

$$Y^{\tau-} = \left(\Phi(Y^{\tau-}) + \sum_{i=1}^{m} F_i(Y^{\tau-}).M_i \right)^{\tau-} \tag{1.4.19}$$

we have

(3) $\rho' := \lim_{k\to\infty} S_k \wedge \tau \le \rho$ a.s. and $Y^{\rho'-} = X^{\rho'-}$;

(4) $\tau \le \rho$ a.s. on $\left\{\omega : \lim_{k\to\infty} T_k(X) \le \lim_{k\to\infty} S_k\right\}$.

We now let $1 \le p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ be two sequences of positive real numbers, $\{S_k\}$ be a sequence of stopping times such that $S_k \uparrow \infty$ a.s. Denote by $\underline{M}^p(\{a_k\}, \{K_{1k}\}, \{S_k\})$ the family of all maps $F : \mathcal{M}_n \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$F(X)\, \mathbb{1}_{]\!]0,T]\!]} = F(X^{T-})\, \mathbb{1}_{]\!]0,T]\!]} \tag{1.4.20}$$

(2) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\left\| \left(F(X^{T_k(X)-}) - F(Y^{T_k(X)-}) \right) \mathbb{1}_{]\!]0,S_k]\!]} \right\|_{\mathcal{P}^p} \le a_k \left\| X - Y \right\|_{\mathcal{H}^p} \tag{1.4.21}$$

(3) for any $X \in \mathcal{M}_n$,

$$\left\| F(X)\, \mathbb{1}_{]\!]0,S_k]\!]} \right\|_{\mathcal{P}^p} \le K_{1k}\left(1 + \left\| X \right\|_{\mathcal{H}^p}\right) \tag{1.4.22}$$

Let $1 \le p < \infty$. Let $\{\beta_k\}$, $\{\alpha_k\}$, $\{K_{2k}\}$ be three sequences of nonnegative real numbers with $0 \le \beta_k < 1$ and $0 \le \alpha_k < 1$. Let $\{S_k\}$ be a sequence of stopping times such that $S_k \uparrow \infty$ a.s. Denote by $\underline{N}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$ the family of all maps $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ such that

(i) for any $X \in \mathcal{M}_n$ and stopping time T,

$$\Phi(X)\, \mathbb{1}_{[\![0,T[\![} = \Phi(X^{T-})\, \mathbb{1}_{[\![0,T[\![} \tag{1.4.23}$$

(ii) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\left\| \left(\Phi(X^{T_k(X)-}) - \Phi(Y^{T_k(X)-}) \right) \mathbb{1}_{]\!]0,S_k]\!]} \right\|_{\mathcal{P}^p} \le \beta_k \left\| X - Y \right\|_{\mathcal{H}^p} \tag{1.4.24}$$

(iii) for any $X \in \mathcal{M}_n$,

$$\left\| \Phi(X)\, \mathbb{1}_{]\!]0,S_k]\!]} \right\|_{\mathcal{H}^p} \le K_{2k} + \alpha_k \left\| X \right\|_{\mathcal{H}^p} \tag{1.4.25}$$

We now have another theorem which can be proved in the same way as Theorem 1.4.2.

Theorem 1.4.4. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $F_i \in \underline{M}^p(\{a_k\}, \{K_{1k}\}, \{S_k\})$, $\Phi \in \underline{N}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$, where $1 \le p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ are two sequences of positive real numbers, $\{\beta_k\}$, $\{\alpha_k\}$, $\{K_{2k}\}$ are three sequences of nonnegative real numbers with $0 \le \beta_k < 1$ and $0 \le \alpha_k < 1$, and $\{S_k\}$ is a sequence of stopping times such that $S_k \uparrow \infty$ a.s.. Then there exists a unique solution in $\mathcal{M}_n$ to Eq.(1.1.2.).

1.5. APPLICATIONS TO DOLEANS-DADE'S EQUATIONS

In this section we will use these theorems obtained in Section 1.3 and 1.4 to study the existence and uniqueness of solutions to Doléans-Dade's equation

$$X(t) = H(t) + \sum_{i=1}^{m} \int_0^t f_i(\cdot, s, X(s-))\, dM_i(s) \tag{1.5.1}$$

Let us begin with the proof of the following theorem.

Theorem 1.5.1. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let f_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \to R^n$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ a sequence of nondecreasing stopping times which satisfy the following conditions:

(1) for every $k = 1, 2, \ldots,$ $1 \le i \le m$, $t \ge 0$ and $x, y \in R^n$ with $|x| \vee |y| \le k$,
$$|(f_i(\omega, t, x) - f_i(\omega, t, y))\, 1\!\!1_{[\![0, S_k]\!]}(t)| \le a_k |x - y| \quad \text{a.s.}$$

(2) for any $x \in R^n$, $f_i(.,.,x)$ is a cadlag. adapted process.

Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k\to\infty} T_k(X) \wedge S_k = \rho$ a.s.;

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{X}$ and

$$X(\sigma) = H(\sigma) + \sum_{i=1}^{m} \int_0^{\sigma} f_i(\cdot, s, X(s-))\, dM_i(s) \tag{1.5.2}$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $X^{\tau-} \in \mathcal{X}$ and

$$Y(t) = H(t) + \sum_{i=1}^{m} \int_0^t f_i(\cdot, s, Y(s-))\, dM_i(s), \quad 0 \le t \le \tau \tag{1.5.3}$$

we have

(3) $\rho' := \lim_{k\to\infty} S_k \wedge \tau \le \rho$ a.s. and $Y^{\rho'-} = X^{\rho'-}$;

(4) $\tau \le \rho$ a.s. on $\left\{\omega : \lim_{k\to\infty} T_k(X) \le \lim_{k\to\infty} S_k\right\}$.

Proof. Without loss of any generality, we assume $M_i(0) = 0$ for all $1 \le i \le m$.

Define $F_i : \mathcal{X} \to \mathcal{X}$ as follows:

$$F_i(X)(t) = f_i(., t, X(t-))$$

then Eq.(1.5.1) can be expressed as

$$X = H + \sum_{i=1}^{m} F_i(X).M_i \tag{1.5.4}$$

It is easy to check that $F_i \in L^p_{M_i}(\{a_k\}, \{S_k\})$ for $1 \le i \le k$. An application of Theorem 1.4.1 implies the desired conclusions and the proof is complete.

Theorem 1.5.2. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let f_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \to R^n$ be random functions, $\{a_k\}$ and $\{K_k\}$ be two sequences of positive real numbers, and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. which satisfy the following conditions:

(1) for every $k = 1, 2, \ldots,$ $1 \le i \le m,$ $t \ge 0$ and $x, y \in R^n$ with $|x| \vee |y| \le k$,

$$|(f_i(\omega, t, x) - f_i(\omega, t, y))\, 1\!\!1_{[\![0, S_k[\![}(t)| \le a_k |x - y| \quad \text{a.s.}$$

(2) for every $k = 1, 2, \ldots,$ $1 \le i \le m,$ $t \ge 0$ and $x \in R^n$,

$$|f_i(\omega, t, x)\, 1\!\!1_{[\![0, S_k[\![}(t)| \le K_k(1 + |x|) \quad \text{a.s.}$$

(3) for any $x \in R^n$, $1 \le i \le m$, $f_i(\cdot, \cdot, x)$ is a cadlag. adapted process.

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.5.1).

Proof. Define F_i as in the proof of Theorem 1.5.1 and stopping times

$$\tau_k = \inf\{t \ge 0 : |H(t)| \ge k\}, \quad k = 1, 2, \ldots$$

Set $U_k = S_k \wedge \tau_k$ and it is clear that $U_k \uparrow \infty$ a.s. We can easily check the assumptions (replace S_k by U_k) in Theorem 1.4.2 are all satisfied. Hence, there exists a unique solution in $\mathcal{X}$ to Eq.(1.5.4), i.e., Eq.(1.5.1) which completes the proof.

As an application of this theorem, we can get the following very useful corollaries.

Corollary 1.5.3. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let f_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \to R^n$ be random functions, $A : \Omega \times R_+ \to R_+$ be right continuous adapted processes which satisfy the following conditions:

(1) for every $1 \le i \le m$, $t \ge 0$ and $x, y \in R^n$,

$$|(f_i(\omega, t, x) - f_i(\omega, t, y))| \le A(\omega, t)|x - y| \quad \text{a.s.}$$

(2) for any $x \in R^n$, $1 \le i \le m$, $f_i(\cdot, \cdot, x)$ is a cadlag. adapted process.

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.5.1).

Proof. Define, for $k = 1, 2, \ldots$, the stopping times

$$\sigma_k = \inf\{ t \geq 0: A(\omega, t) \geq k \}$$

$$\theta_k = \inf\{ t \geq 0: f_i(\omega, t, 0) \geq k \text{ for some } i = 1, \ldots, m \}$$

and set $S_k = \sigma_k \wedge \theta_k$. Clearly, $S_k \uparrow \infty$ a.s. and f_i $(1 \leq i \leq m)$ satisfy all of the conditions in Theorem 1.5.2. In fact, we only need to check condition (2) in Theorem 1.5.2 as follows: for every $k = 1, 2, \ldots$, $1 \leq i \leq m$, $t \geq 0$ and $x \in R^n$,

$$\begin{aligned} &| f_i(\omega, t, x) \mathbb{1}_{[\![0, S_k]\!]}(t) | \\ &\leq | (f_i(\omega, t, x) - | f_i(\omega, t, 0)) \mathbb{1}_{[\![0, S_k]\!]}(t) | + | f_i(\omega, t, 0) \mathbb{1}_{[\![0, S_k]\!]}(t) | \\ &\leq k | x | + k = k (1 + | x |) \quad \text{a.s.} \end{aligned}$$

Hence, by Theorem 1.5.2, we get desired conclusion immediately.

Corollary 1.5.4. Under the assumptions of Corollary 1.5.3 with $A(\omega, t)$ replaced by $a(\omega)$, where $a(\cdot) : \Omega \to R_+$ is a random variable (i.e., $\mathcal{F}$-measurable), then conclusion of Corollary 1.5.3 still holds.

Proof. It is obvious that there exists $\Omega^* \in \mathcal{F}$ with $P(\Omega^*) = 1$ such that

$$| (f_i(\omega, t, x) - f_i(\omega, t, y)) | \leq a(\omega) | x - y |$$

for all $\omega \in \Omega^*$, $t \geq 0$, $x, y \in R^n$ and $1 \leq i \leq m$. Define

$$W(\omega, t) = \begin{cases} 0, & \text{if } \omega \notin \Omega^* \\ \displaystyle\sup_{\substack{0 \leq s \leq t,\, 1 \leq i \leq m \\ x, y \in R^n,\, x \neq y}} \frac{| f_i(\omega, s, x) - f_i(\omega, s, y) |}{| x - y |}, & \text{if } \omega \in \Omega^* \end{cases}$$

which is an adapted process with

$$W(\omega, t) \leq W(\omega, s) \leq a(\omega) \qquad \text{for all } 0 \leq t \leq s < \infty,\ \omega \in \Omega$$

Define

$$A(\omega, t) = \lim_{s \downarrow t} W(\omega, s) \qquad \text{for } (\omega, t) \in \Omega \times R_+$$

Then A is a right continuous adapted process and we have

$$| (f_i(\omega, t, x) - f_i(\omega, t, y)) | \leq A(\omega, t) | x - y | \quad \text{a.s.}$$

for all $t \geq 0$, $x, y \in R^n$ and $1 \leq i \leq m$. Hence, the assumptions of Corollary 1.5.3 hold and we get the conclusion.

Using Corollary 1.5.4, in the almost same way as the proof of Theorem 1.3.3 and 1.4.2, we can prove the following more general theorem.

Theorem 1.5.5. Under the assumptions of Theorem 1.5.2 with $\{a_k\}$ replaced by $\{a_k(\omega)\}$, the sequences of positive random variables, the assertion of Theorem 1.5.2 still holds.

1.6. APPLICATIONS TO DELAY SIES

Mohammed [1] studied systematically delay stochastic differential equations with respect to Brownian motion. In this section we will use the results in Sections 1.3 and 1.4 to discuss the existence and uniqueness theorems for a more general delay SIES

$$X = \Psi(X, X^d) + \sum_{i=1}^{m} G_i(X, X^d).M_i \tag{1.6.1}$$

where $X^d = (X^d(t))$ is said to be the delay process of X defined as follows:

$$X^d(t) = X(t-d(t))\,\mathbb{1}_{R_+-D}(t) + A(t-d(t))\,\mathbb{1}_D(t)$$

in which $d(t) : R_+ \to R_+$ is a continuous function, $D = \{t : t - d(t) < 0,\ t \geq 0\}$ and $A : \Omega \times \hat{D} \to R^n$, $\hat{D} = \{t-d(t) : t \in D\}$. We here allow the time lag d(t) depends on t. Our method is different from Mohammed [1]. Throughout the remainder of this Chapter we assume

(H1) A(.) is a continuous process on $\hat{D}$ which is $\mathcal{F}_0$-measurable,
(H2) t-d(t) is nondecreasing on $t \in D$.

Let $\mathcal{B}(B)$ stand for the set of all Borel subset of $B(\subset R)$. We denote by $\mathcal{W}$ the set of all n-dimensional progressive processes. We also define $\mathcal{Q} = \{X \in \mathcal{W} : X$ is a left limit process, i.e., there exists the finite limit $\lim_{k\to\infty} X(t_k)$ a.s. for any increasing sequence $\{t_k\}$ in $R_+\}$. We shall need the following lemmas.

Lemma 1.6.1. $X^d \in \mathcal{W}$ if $X \in \mathcal{W}$.

Proof. Let $X \in \mathcal{W}$. For any $T \geq 0$ and Borel subset B of R^n, we have

$$\begin{aligned} &\{(\omega, t) : X^d(\omega, t) \in B,\ 0 \leq t \leq T\} \\ &= \{(\omega, t) : X(\omega, t-d(t)) \in B,\ t \in [0, T] \cap (R_+-D)\} \\ &\quad \cup \{(\omega, t) : X(\omega, t-d(t)) \in B,\ t \in [0, T] \cap D\} \end{aligned} \tag{1.6.2}$$

However, by the assumptions, it is clear that

$$\{(\omega,t): X(\omega,t-d(t)) \in B,\ t \in [0,T] \cap D\}$$
$$\in \mathcal{F}_T \times \mathcal{B}([0,T] \cap D) \subset \mathcal{F}_T \times \mathcal{B}([0,T]) \tag{1.6.3}$$

In addition,

$$X(\omega,t): (\Omega \times [0,T],\ \mathcal{F}_T \times \mathcal{B}([0,T]) \to (R^n,\ \mathcal{B}(R^n))$$

and

$$(\omega, t-d(t)): (\Omega \times [0,T] \cap (R_+-D),\ \mathcal{F}_T \times \mathcal{B}([0,T] \cap (R_+-D)))$$
$$\to (\Omega \times [0,T],\ \mathcal{F}_T \times \mathcal{B}([0,T])) \quad .$$

Hence

$$X(\omega, t-d(t)): (\Omega \times [0,T] \cap (R_+-D),\ \mathcal{F}_T \times \mathcal{B}([0,T] \cap (R_+-D)))$$
$$\to (R^n,\ \mathcal{B}(R^n))$$

which deduce

$$\{(\omega,t): X(\omega,t-d(t)) \in B,\ t \in [0,T] \cap (R_+-D)\}$$
$$\in \mathcal{F}_T \times \mathcal{B}([0,T] \cap (R_+-D)) \subset \mathcal{F}_T \times \mathcal{B}([0,T]) \quad . \tag{1.6.4}$$

Combining (1.6.2) - (1.6.4), we get

$$\{(\omega,t): X^d(\omega,t) \in B,\ 0 \le t \le T\} \in \mathcal{F}_T \times \mathcal{B}([0,T])$$

which implies the assertion $X^d \in \mathcal{W}$.

Lemma 1.6.2. If $X \in \mathcal{Q}$, then $X^d \in \mathcal{Q}$.

We omit the proof since it is easy.

Lemma 1.6.3. If $X \in \mathcal{X}$ and T is a stopping time, we then have

$$(X^d)^{T-} = ((X^{T-})^d)^{T-} \tag{1.6.5}$$

and

$$(X^d)^T = ((X^T)^d)^T \tag{1.6.6}$$

Proof. For all $t \ge 0$, we have

$$((X^{T-})^d \mathbb{1}_{[\![0,T[\![})(t)$$
$$= \left\{X(t-d(t))\, \mathbb{1}_{[\![0,T[\![}(t-d(t)) + X(T-)\, \mathbb{1}_{[\![T,\infty[\![}(t-d(t))\right\} \mathbb{1}_{R_+-D}(t)\, \mathbb{1}_{[\![0,T[\![}(t)$$
$$+ A(t-d(t))\, \mathbb{1}_D(t)\, \mathbb{1}_{[\![0,T[\![}(t)$$

However, $0 \le t - d(t) \le t < T$ if $t \in R_+ - D$ and $t < T$. Hence

$$
\begin{aligned}
& ((X^{T-})^d \mathbb{1}_{[\![0,T[\![})(t) \\
= & \left\{ X(t-d(t))\, \mathbb{1}_{R_+-D}(t) + A(t-d(t))\, \mathbb{1}_D(t) \right\} \mathbb{1}_{[\![0,T[\![}(t) \\
= & (X^d \mathbb{1}_{[\![0,T[\![})(t)
\end{aligned}
$$

i.e.,

$$(X^{T-})^d \mathbb{1}_{[\![0,T[\![} = X^d \mathbb{1}_{[\![0,T[\![}$$

which, together with Lemma 1.6.2, yields (1.6.5).

Similarly, we have

$$
\begin{aligned}
& ((X^T)^d \mathbb{1}_{[\![0,T]\!]})(t) \\
= & \left\{ X(t-d(t))\, \mathbb{1}_{[\![0,T]\!]}(t-d(t)) + X(T)\, \mathbb{1}_{]\!]T,\infty[\![}(t-d(t)) \right\} \mathbb{1}_{R_+-D}(t)\, \mathbb{1}_{[\![0,T]\!]}(t) \\
& + A(t-d(t))\, \mathbb{1}_D(t)\, \mathbb{1}_{[\![0,T]\!]}(t) \\
= & \left\{ X(t-d(t))\, \mathbb{1}_{R_+-D}(t) + A(t-d(t))\, \mathbb{1}_D(t) \right\} \mathbb{1}_{[\![0,T]\!]}(t) \\
= & (X^d \mathbb{1}_{[\![0,T]\!]})(t)
\end{aligned}
$$

which implies (1.6.6) immediately.

We now introduce some notations.

Let $M \in \mathcal{M}$, $1 \le p < \infty$, $a > 0$. Denote by $\mathcal{W}^p_M(a)$ the family of all maps $G : \mathcal{X} \times \mathcal{Q} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$G(X, X^d)\, \mathbb{1}_{]\!]0,T]\!]} = G(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{]\!]0,T]\!]},$$

(2) for any $X, Y \in \mathcal{X}$ and $U, V \in \mathcal{Q}$,

$$\| G(X,U) - G(Y,V) \|_{\varphi^p} \le a\left(\| X - Y \|_{\varphi^p} + \| U - V \|_{\varphi^p} \right),$$

(3) $G(0,0) \in \mathcal{L}(M)$.

Let $1 \le p < \infty$ and $0 \le \beta < 1/2$. Denote also by $\mathcal{V}^p(\beta)$ the family of all maps $\Psi : \mathcal{X} \times \mathcal{Q} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Psi(X, X^d)\, \mathbb{1}_{[\![0,T[\![} = \Psi(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{[\![0,T[\![},$$

(ii) for any $X, Y \in \mathcal{X}$,

$$\| \Psi(X, X^d) - \Psi(Y, Y^d) \|_{\varphi^p} \le \beta\left(\| X - Y \|_{\varphi^p} + \| X^d - Y^d \|_{\varphi^p} \right)$$

Before we state our first existence and uniqueness theorem for Eq.(1.6.1), we still need some lemmas.

Lemma 1.6.4 (Yan [1]). Let $M \in \mathcal{M}$ and H be a local φ^p-predictable process. Then $H \in \mathcal{L}(M)$.

Lemma 1.6.5. Let $M \in \mathcal{M}$, $1 \le p < \infty$, $a > 0$. If $F \in \mathcal{W}^p_M(a)$, then $F(X, X^d) \in \mathcal{L}(M)$ for all $X \in \mathcal{X}$.

Proof. Let $X \in \mathcal{X}$. Define stopping times

$$T_k = \inf\{ t : |X(t)| \ge k \text{ or } |X^d(t)| \ge k,\ t \ge 0 \}, \qquad k = 1, 2, \ldots$$

Then $\| X^{T_k-} \|_{\varphi^p} \le k$ and $\| (X^d)^{T_k-} \|_{\varphi^p} \le k$. Using the assumptions, we have

$$\begin{aligned}
&\| (F(X, X^d) - F(0,0)) \mathbb{1}_{]\!]0, T_k]\!]} \|_{\varphi^p} \\
&= \| (F(X^{T_k-}, (X^d)^{T_k-}) - F(0,0)) \mathbb{1}_{]\!]0, T_k]\!]} \|_{\varphi^p} \\
&\le \| (F(X^{T_k-}, (X^d)^{T_k-}) - F(0,0)) \|_{\varphi^p} \\
&\le a (\| X^{T_k-} \|_{\varphi^p} + \| (X^d)^{T_k-} \|_{\varphi^p}) \\
&\le 2ak
\end{aligned}$$

which yields that $F(X, X^d) - F(0,0)$ is a local φ^p-predictable process since $T_k \to \infty$ a.s. Hence, by Lemma 1.6.4, we obtain $F(X, X^d) - F(0,0) \in \mathcal{L}(M)$ which implies $F(X, X^d) \in \mathcal{L}(M)$ since $F(0,0) \in \mathcal{L}(M)$. The proof is complete.

We now can prove the following theorem.

Theorem 1.6.6. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \mathcal{W}^p_{M_i}(a)$, $\Psi \in \mathcal{V}^p(\beta)$, where $1 \le p < \infty$, $a > 0$, $0 \le \beta < 1/2$. Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.6.1).

Proof. By Lemma 1.6.5, Eq.(1.6.1) is well defined. Define

$$\Phi(X) = \Psi(X, X^d) \quad \text{and} \quad F_i(X) = G_i(X, X^d)$$

for $X \in \mathcal{X}$ and $1 \le i \le m$. Then Eq.(1.6.1) is equivalent to the equation

$$X = \Phi(X) + \sum_{i=1}^{m} F_i(X).M_i \tag{1.6.7}$$

We now claim that $F_i \in \mathcal{L}^p_{M_i}(2a)$. Indeed,

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F_i(X)\, \mathbb{1}_{]\!]0,T]\!]} = G_i(X, X^d)\, \mathbb{1}_{]\!]0,T]\!]} = G_i(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{]\!]0,T]\!]}$$
$$= G_i(X^{T-}, ((X^{T-})^d)^{T-})\, \mathbb{1}_{]\!]0,T]\!]} = G_i(X^{T-}, (X^{T-})^d)\, \mathbb{1}_{]\!]0,T]\!]}$$
$$= F_i(X^{T-})\, \mathbb{1}_{]\!]0,T]\!]}$$

where Lemma 1.6.3 has been used.

(2) for any $X, Y \in \mathcal{X}$;

$$\| F_i(X) - F_i(Y) \|_{\varphi^p} \le \| G_i(X, X^d) - G_i(Y, Y^d) \|_{\varphi^p}$$

$$\le a (\|X - Y\|_{\varphi^p} + \|X^d - Y^d\|_{\varphi^p}) \le 2a \|X - Y\|_{\varphi^p}$$

(3) by Lemma 1.6.5, $F_i(0) = F_i(0, 0^d) \in \mathcal{L}(M)$.

Similarly, we can verify $\Phi \in \mathcal{C}^p(2\beta)$. In view of Theorem 1.2.1, we get the desired conclusion which completes the proof.

Let $M \in \mathcal{M}$, $1 \le p < \infty$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s.. Denote by $\mathcal{W}^p_M(\{a_k\}, \{S_k\})$ the family of all maps $G : \mathcal{X} \times \mathcal{Q} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$G(X, X^d)\, \mathbb{1}_{]\!]0,T]\!]} = G(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{]\!]0,T]\!]}$$

(2) for any $X, Y \in \mathcal{X}$ and $U, V \in \mathcal{Q}$ and $k = 1, 2, \ldots,$

$$\| (G(X, U) - G(Y, V))\, \mathbb{1}_{]\!]0,S_k]\!]} \|_{\varphi^p} \le a_k (\|X - Y\|_{\varphi^p} + \|U - V\|_{\varphi^p})$$

(3) $G(0, 0) \in \mathcal{L}(M)$.

Let $1 \le p < \infty$, $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1/2$ and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote also by $\mathcal{V}^p(\{\beta_k\}, \{S_k\})$ the family of all maps $\Psi : \mathcal{X} \times \mathcal{Q} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Psi(X, X^d)\, 1\!\!1_{[\![0,T[\![} = \Psi(X^{T-}, (X^d)^{T-})\, 1\!\!1_{[\![0,T[\![}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| (\Psi(X, X^d) - \Psi(Y, Y^d))^{S_k-} \|_{\varphi^p} \leq \beta_k(\| X - Y \|_{\varphi^p} + \| X^d - Y^d \|_{\varphi^p})$$

The following theorem is a generalization of Theorem 1.6.6.

Theorem 1.6.7. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \mathcal{W}^p_M(\{a_k\}, \{S_k\})$, $\Psi \in \mathcal{V}^p(\{\beta_k\}, \{S_k\})$, where $1 \leq p < \infty$, $\{a_k\}$ is a sequence of positive real numbers and $\{\beta_k\}$ is a sequence of real numbers with $0 \leq \beta_k < 1/2$ and $\{S_k\}$ is a sequence of stopping times with $S_k \uparrow \infty$ a.s.. Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.6.1).

Proof. We can easily, in the same way as the proof of Lemma 1.6.5, prove that $G_i(X, X^d) \in \mathcal{L}(M_i)$ for all $X \in \mathcal{X}$. Hence Eq.(1.6.1) is well defined. For each $k = 1, 2, \ldots,$ we can also easily verify that $G_i(X, X^d)\, 1\!\!1_{]\!]0, S_k]\!]} \in \mathcal{W}^p_{M_i^{S_k-}}(a_k)$ and $\Psi(X, X^d)^{S_k-} \in \mathcal{V}^p(\beta_k)$. Hence, by Theorem 1.6.5, there is a unique solution X^k in $\mathcal{X}$ to the equation

$$X = \Psi(X, X^d)^{S_k-} + \sum_{i=1}^m G_i(X, X^d).M_i^{S_k-}$$

We now claim

$$(X^{k+1})^{S_k-} = (X^k)^{S_k-} \quad \text{a.s.} \qquad \text{for all } k = 1, 2, \ldots, \tag{1.6.8}$$

In fact, we have

$$(X^{k+1})^{S_k-} = \Psi(X^{k+1}, (X^{k+1})^d)^{S_k-} + \sum_{i=1}^m G_i(X^{k+1}, (X^{k+1})^d).M_i^{S_k-}$$

$$= \Psi((X^{k+1})^{S_k-}, ((X^{k+1})^d)^{S_k-})^{S_k-} + \sum_{i=1}^m G_i((X^{k+1})^{S_k-}, ((X^{k+1})^d)^{S_k-})\, 1\!\!1_{]\!]0, S_k]\!]} \cdot M_i^{S_k-}$$

$$= \Psi\big((X^{k+1})^{S_k-}, (((X^{k+1})^{S_k-})^d)^{S_k-}\big)^{S_k-}$$

$$+ \sum_{i=1}^m G_i\big((X^{k+1})^{S_k-}, (((X^{k+1})^{S_k-})^d)^{S_k-}\big)\, 1\!\!1_{]\!]0, S_k]\!]} \cdot M_i^{S_k-}$$

$$= \Psi\big((X^{k+1})^{S_k-}, ((X^{k+1})^{S_k-})^d\big)^{S_k-} + \sum_{i=1}^{m} G_i\big((X^{k+1})^{S_k-}, ((X^{k+1})^{S_k-})^d\big) . M_i^{S_k-}$$

where Lemma 1.6.3 has been used. Similarly, we also have

$$(X^k)^{S_k-} = \Psi\big((X^k)^{S_k-}, ((X^k)^{S_k-})^d\big)^{S_k-} + \sum_{i=1}^{m} G_i\big((X^k)^{S_k-}, ((X^k)^{S_k-})^d\big) . M_i^{S_k-}$$

Hence, by Theorem 1.6.6, we deduce (1.6.8). Thus, there exists a unique X in $\mathcal{X}$ such that $X^{S_k-} = (X^k)^{S_k-}$ for all k = 1, 2, ... , and we can similarly prove

$$X^{S_k-} = (X^k)^{S_k-} =$$

$$= \Psi\big((X^k)^{S_k-}, (((X^k)^{S_k-})^d)^{S_k-}\big)^{S_k-}$$

$$+ \sum_{i=1}^{m} G_i\big((X^k)^{S_k-}, (((X^k)^{S_k-})^d)^{S_k-}\big) \mathbb{1}_{]\!]0, S_k]\!]} \cdot M_i^{S_k-}$$

$$= \Psi\big(X^{S_k-}, ((X^{S_k-})^d)^{S_k-}\big)^{S_k-} + \sum_{i=1}^{m} G_i\big(X^{S_k-}, ((X^{S_k-})^d)^{S_k-}\big) \mathbb{1}_{]\!]0, S_k]\!]} \cdot M_i^{S_k-}$$

$$= \Big(\Psi(X, X^d) + \sum_{i=1}^{m} G_i(X, X^d).M_i \Big)^{S_k-}$$

which means X is a solution of Eq.(1.6.1). Uniqueness is obvious. So we have completed the proof.

Let $M \in \mathcal{M}$, $1 \leq p < \infty$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ a sequence of nondecreasing stopping times. Denote by $\underline{W}_M^p(\{a_k\}, \{S_k\})$ the family of all maps $G : \mathcal{X} \times \mathcal{Q} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$G(X, X^d)\, \mathbb{1}_{]\!]0, T]\!]} = G(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{]\!]0, T]\!]}$$

(2) for any $X, Y \in \mathcal{X}$ and $U, V \in \mathcal{Q}$ and k = 1, 2, ...,

$$\| (G(X^{[k]}, U^{[k]}) - G(Y^{[k]}, V^{[k]}))\, \mathbb{1}_{]\!]0, S_k]\!]} \|_{\varphi^p}$$

$$\leq a_k (\| X - Y \|_{\varphi^p} + \| U - V \|_{\varphi^p})$$

(3) $G(0, 0)\, \mathbb{1}_{]\!]0, S_k]\!]} \in \mathcal{L}(M)$ for all k = 1, 2, ...

Let $1 \le p < \infty$, $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1/2$ and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s.. Denote also by $\underline{V}^p(\{\beta_k\}, \{S_k\})$ the family of all maps $\Psi : \mathcal{X} \times \mathcal{Q} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Psi(X, X^d)\,\mathbb{1}_{[\![0,T[\![} = \Psi(X^{T-}, (X^d)^{T-})\,\mathbb{1}_{[\![0,T[\![}$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| (\Psi(X^{[k]}, (X^{[k]})^d) - \Psi(Y^{[k]}, (Y^{[k]})^d))^{S_k-} \|_{\varphi^p}$$

$$\le \beta_k(\|X - Y\|_{\varphi^p} + \|X^d - Y^d\|_{\varphi^p})$$

The following theorem can be proved in the same way as the proof of Theorem 1.6.6 by using Theorem 1.4.1.

Theorem 1.6.8. Let M_i ($1 \le i \le m$) $\in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \underline{W}^p_{M_i}(\{a_k\}, \{S_k\})$, $\Psi \in \underline{V}^p(\{\beta_k\}, \{S_k\})$, where $1 \le p < \infty$, $\{a_k\}$ is a sequence of positive real numbers and $\{\beta_k\}$ a sequence of real numbers with $0 \le \beta_k < 1/2$ and $\{S_k\}$ a sequence of nondecreasing stopping times. Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k\to\infty} T_k(X) \wedge S_k = \rho$ a.s.;

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{X}$ and

$$X^{\sigma-} = \Big(\Phi(X^{\sigma-}, (X^d)^{\sigma-}) + \sum_{i=1}^{m} F_i(X^{\sigma-}, (X^d)^{\sigma-}).M_i \Big)^{\sigma-} \qquad (1.6.9)$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $X^{\tau-} \in \mathcal{X}$ and

$$Y^{\tau-} = \Big(\Phi(Y^{\tau-}, (Y^d)^{\tau-}) + \sum_{i=1}^{m} F_i(Y^{\tau-}, (Y^d)^{\tau-}).M_i \Big)^{\tau-} \qquad (1.6.10)$$

we have

(3) $\rho' := \lim_{k\to\infty} S_k \wedge \tau \le \rho$ a.s. and $Y^{\rho'-} = X^{\rho'-}$;

(4) $\tau \le \rho$ a.s. on $\Big\{ \omega : \lim_{k\to\infty} T_k(X) \le \lim_{k\to\infty} S_k \Big\}$.

Let $1 \le p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$be sequences of positive real numbers, and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote by $\mathcal{W}^p(\{a_k\}, \{K_{1k}\})$ the set of all maps $G : \mathcal{X} \times \mathcal{Q} \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$G(X, X^d)\, \mathbb{1}_{\rrbracket 0, T \rrbracket} = G(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{\rrbracket 0, T \rrbracket}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| \left(G((X^{[k]}), (X^{[k]})^d) - G(Y^{[k]}, (Y^{[k]})^d) \right) \mathbb{1}_{\rrbracket 0, S_k \rrbracket} \|_{\varphi^p}$$

$$\leq a_k \left(\| X^{[k]} - Y^{[k]} \|_{\varphi^p} + \| (X^{[k]})^d - (X^{[k]})^d \|_{\varphi^p} \right)$$

(3) for any $X \in \mathcal{X}$ and $U \in \mathcal{Q}$,

$$\| G(X, U)\, \mathbb{1}_{\rrbracket 0, S_k \rrbracket} \|_{\varphi^p} \leq K_{1k}(1 + \| X \|_{\varphi^p} + \| U \|_{\varphi^p})$$

Let $1 \leq p < \infty$, $\{\beta_k\}$, $\{\alpha_k\}$ and $\{K_{2k}\}$ be sequences of nonnegative real numbers with $0 \leq \beta_k < 1/2$ and $0 \leq \alpha_k < 1/2$, $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote by $\mathcal{V}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\})$ the family of all maps $\Psi : \mathcal{X} \times \mathcal{Q} \to \mathcal{X}$ such that

(i) for any $X \in \mathcal{X}$ and stopping time T,

$$\Psi(X, X^d)\, \mathbb{1}_{\llbracket 0, T \llbracket} = \Psi(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{\llbracket 0, T \llbracket} .$$

(ii) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| \left(\Psi((X^{[k]}), (X^{[k]})^d) - \Psi(Y^{[k]}, (Y^{[k]})^d) \right)^{S_k-} \|_{\varphi^p}$$

$$\leq \beta_k \left(\| X^{[k]} - Y^{[k]} \|_{\varphi^p} + \| (X^{[k]})^d - (X^{[k]})^d \|_{\varphi^p} \right)$$

(iii) for any $X \in \mathcal{X}$,

$$\| (\Psi(X, X^d))^{S_k-} \|_{\varphi^p} \leq K_{2k} + \alpha_k (\| X \|_{\varphi^p} + \| X^d \|_{\varphi^p})$$

Theorem 1.6.9. Let M_i $(1 \leq i \leq m) \in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \mathcal{W}^p(\{a_k\}, \{K_{1k}\})$, $\Psi \in \mathcal{V}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\})$, where $1 \leq p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ are sequences of positive real numbers, $\{\beta_k\}$, $\{\alpha_k\}$ and $\{K_{2k}\}$ are sequences of nonnegative real numbers with $0 \leq \beta_k < 1/2$ and $0 \leq \alpha_k < 1/2$, $\{S_k\}$ is a sequence of stopping times with $S_k \uparrow \infty$ a.s. Assume

$$\left(E \sup_{t \in D} | A(t-d(t)) |^p \right)^{1/p} = K_3 < \infty$$

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.6.1).

Proof. We first claim that $G_i(X, X^d) \in \mathcal{L}(M_i)$ for all $X \in \mathcal{X}$ and $1 \leq i \leq m$ so that Eq.(1.6.1) is well defined. Indeed, let $X \in \mathcal{X}$ and define stopping times

$$T_k = \inf\{ t : |X(t)| \geq k \text{ or } |X^d(t)| \geq k,\ t \geq 0 \}, \qquad k = 1, 2, \ldots$$

Then $\| X^{T_k-} \|_{\varphi^p} \leq k$ and $\| (X^d)^{T_k-} \|_{\varphi^p} \leq k$. Using the assumptions, we have

$$\| G_i(X, X^d) 1\!\!1_{]\!]0, T_k \wedge S_k]\!]} \|_{\varphi^p} = \| G_i(X^{T_k-}, (X^d)^{T_k-}) 1\!\!1_{]\!]0, T_k \wedge S_k]\!]} \|_{\varphi^p}$$

$$\leq \| G_i(X^{T_k-}, (X^d)^{T_k-}) 1\!\!1_{]\!]0, T_k \wedge S_k]\!]} \|_{\varphi^p}$$

$$\leq K_{1k} (1 + \| X^{T_k-} \|_{\varphi^p} + \| (X^d)^{T_k-} \|_{\varphi^p}) \leq K_{1k} (1 + 2k)$$

Hence, by Lemma 1.6.4, we get $G_i(X, X^d) \in \mathcal{L}(M_i)$.

We now define F_i $(1 \leq i \leq m)$ and Φ as in the proof of Theorem 1.6.6, and then Eq.(1.6.1) is equivalent to Eq.(1.6.7). We can check $F_i \in \underline{L}^p(\{2a_k\}, \{K_{4k}\}, \{S_k\})$, where $K_{4k} = K_{1k}(1 + K_3) \vee 2K_{1k}$. In fact, in the same way as the proof of Theorem 1.6.6, we can show

(1) for any $X \in \mathcal{X}$ and stopping time T,

$$F_i(X) 1\!\!1_{]\!]0, T]\!]} = F_i(X^{T-}) 1\!\!1_{]\!]0, T]\!]}$$

(2) for any $X, Y \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| F_i(X^{[k]}) - F(Y^{[k]}) 1\!\!1_{]\!]0, S_k]\!]} \|_{\varphi^p} \leq 2a_k \| X - Y \|_{\varphi^p}$$

We also have

(3) for any $X \in \mathcal{X}$ and $k = 1, 2, \ldots,$

$$\| F_i(X) 1\!\!1_{]\!]0, S_k]\!]} \|_{\varphi^p} = \| G_i(X, X^d) 1\!\!1_{]\!]0, S_k]\!]} \|_{\varphi^p}$$

$$\leq K_{1k}(1 + \| X \|_{\varphi^p} + \| X^d \|_{\varphi^p})$$

$$\leq K_{1k}(1 + 2 \| X \|_{\varphi^p} + K_3) \leq K_{4k}(1 + \| X \|_{\varphi^p})$$

Similarly, we can also verify $\Phi \in \underline{C}^p(\{2\beta_k\}, \{2\alpha_k\}, \{K_{5k}\}, \{S_k\})$, where $K_{5k} = K_{2k} + \alpha K_3$. Therefore, an application of Theorem 1.4.2 implies the assertion which completes the proof.

In the sequel of this section we assume

(H) $d(\cdot) : R_+ \to R_+$ is a continuous function such that $t - d(t)$ is nonnegative and nondecreasing.

In this case, $X^d(t) = X(t-d(t))$ for all $t \geq 0$. Clearly, we have

$$X^d \in \mathcal{M}_n \text{ if } X \in \mathcal{M}_n \tag{1.6.11}$$

We still need some notations.

Let $1 \le p < \infty$, $M \in \mathcal{M}$, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote by $\mathbb{Q}_M^p(\{a_k\}, \{S_k\})$ the set of all maps $G : \mathcal{M}_n \times \mathcal{M}_n \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$G(X, X^d)\, 1\!\!1_{]\!]0,T]\!]} = G(X^{T-}, (X^d)^{T-})\, 1\!\!1_{]\!]0,T]\!]}$$

(2) for any $X, Y, U, V \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\|(G(X, U) - G(Y, V))\, 1\!\!1_{]\!]0, S_k]\!]}\|_{\mathcal{P}^p} \le a_k(\|X - Y\|_{\mathcal{H}^p} + \|U - V\|_{\mathcal{H}^p})$$

(3) $G(0, 0) \in \mathcal{L}(M)$.

Let $1 \le p < \infty$, $\{\beta_k\}$ be a sequence of real numbers with $0 \le \beta_k < 1/2$ and $\{S_k\}$ a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote by $\mathbb{N}^p(\{\beta_k\}, \{S_k\})$ the family of all maps $\Psi : \mathcal{M}_n \times \mathcal{M}_n \to \mathcal{M}_n$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$\Psi(X, X^d)\, 1\!\!1_{[\![0,T[\![} = \Psi(X^{T-}, (X^d)^{T-})\, 1\!\!1_{[\![0,T[\![}$$

(2) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\|(G(X, X^d) - G(Y, Y^d))^{S_k-}\|_{\mathcal{H}^p} \le \beta_k(\|X - Y\|_{\mathcal{H}^p} + \|X^d - Y^d\|_{\mathcal{H}^p})$$

Let $1 \le p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ be two sequences of positive real numbers, $\{S_k\}$ a sequence of stopping times such that $S_k \uparrow \infty$ a.s. Denote by $\mathcal{Q}^p(\{a_k\}, \{K_{1k}\}, \{S_k\})$ the set of all maps $G : \mathcal{M}_n \times \mathcal{M}_n \to \mathcal{P}$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$G(X, X^d)\, 1\!\!1_{]\!]0,T]\!]} = G(X^{T-}, (X^d)^{T-})\, 1\!\!1_{]\!]0,T]\!]}$$

(2) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\|\left(G(X^{T_k(X)-}, (X^d)^{T_k(X)-}) - G(X^{T_k(X)-}, (X^d)^{T_k(X)-})\right) 1\!\!1_{]\!]0, S_k]\!]}\|_{\mathcal{P}^p}$$

$$\le a_k(\|X - Y\|_{\mathcal{H}^p} + \|X^d - Y^d\|_{\mathcal{H}^p})$$

(3) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots,$

$$\|G(X, Y)\, 1\!\!1_{]\!]0, S_k]\!]}\|_{\mathcal{P}^p} \le K_{2k}(1 + \|X\|_{\mathcal{H}^p} + \|Y\|_{\mathcal{H}^p})$$

Let $1 \le p < \infty$, $\{\beta_k\}$, $\{\alpha_k\}$ and $\{K_{2k}\}$ be sequences of nonnegative real numbers with $0 \le \beta_k < 1/2$ and $0 \le \alpha_k < 1/2$, $\{S_k\}$ a sequence of stopping times with $S_k \uparrow \infty$ a.s. Denote by $\mathcal{N}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$ the family of all maps $\Psi : \mathcal{M}_n \times \mathcal{M}_n \to \mathcal{M}_n$ such that

(1) for any $X \in \mathcal{M}_n$ and stopping time T,

$$\Psi(X, X^d)\, \mathbb{1}_{]\!]0,T]\!]} = \Psi(X^{T-}, (X^d)^{T-})\, \mathbb{1}_{]\!]0,T]\!]}$$

(2) for any $X, Y \in \mathcal{M}_n$ and $k = 1, 2, \ldots$,

$$\| (\Psi(X^{T_k(X)-}, (X^d)^{T_k(X)-}) - \Psi(X^{T_k(X)-}, (X^d)^{T_k(X)-}))^{S_k-} \|_{\mathcal{H}^p}$$

$$\le \beta_k (\| X - Y \|_{\mathcal{H}^p} + \| X^d - Y^d \|_{\mathcal{H}^p})$$

(3) for any $X \in \mathcal{M}_n$ and $k = 1, 2, \ldots$,

$$\| \Psi(X, X^d)^{S_k-} \|_{\mathcal{H}^p} \le K_{2k} + \alpha_k (\| X \|_{\mathcal{H}^p} + \| X^d \|_{\mathcal{H}^p})$$

We can also similarly prove the following theorems (the proofs are omitted).

Theorem 1.6.10. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \mathbb{Q}_M^p(\{a_k\}, \{S_k\})$, $\Psi \in \mathbb{N}^p(\{\beta_k\}, \{S_k\})$, where $1 \le p < \infty$, $\{a_k\}$ and $\{\beta_k\}$ are sequences of real numbers with $a_k > 0$ and $0 \le \beta_k < 1/2$, $\{S_k\}$ is a sequence of stopping times with $S_k \uparrow \infty$ a.s.. Assume (H) holds. Then there exists a unique solution in $\mathcal{M}_n$ to Eq.(1.6.1).

Theorem 1.6.11. Let M_i $(1 \le i \le m) \in \mathcal{M}$ with $M_i(0) = 0$, $G_i \in \mathcal{Q}^p(\{a_k\}, \{K_{1k}\}, \{S_k\})$, $\Psi \in \mathcal{N}^p(\{\beta_k\}, \{\alpha_k\}, \{K_{2k}\}, \{S_k\})$, where $1 \le p < \infty$, $\{a_k\}$ and $\{K_{1k}\}$ are two sequences of positive real numbers, $\{\beta_k\}$, $\{\alpha_k\}$ and $\{K_{2k}\}$ are sequences of nonnegative real numbers with $0 \le \beta_k < 1/2$ and $0 \le \alpha_k < 1/2$, $\{S_k\}$ is a sequence of stopping times with $S_k \uparrow \infty$ a.s.. Assume (H) holds. Then there exists a unique solution in $\mathcal{M}_n$ to Eq.(1.6.1).

1.7. DELAY DOLEANS-DADE'S EQUATIONS

Like section 1.5, we will apply the results in section 1.6 to discuss the existence and uniqueness theorems for the following delay Doléans-Dade's equation

$$X(t) = H(t) + \sum_{i=1}^{m} \int_0^t g_i(\cdot, s, X(s-), X^d(s-)\, dM_i(s) \tag{1.7.1}$$

in this section. The proofs of the following several existence and uniqueness theorems for Eq.(1.7.1) are all omitted since they are similar to those of Theorems in section 1.5 by using the corresponding theorems in section 1.6.

Theorem 1.7.1. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let g_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \times R^n \to R^n$ be random functions, $\{a_k(\omega)\}$ be a sequence of positive random variables and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. which satisfy the following conditions:

(1) for every $k = 1, 2, \;\;, \; 1 \le i \le m, \; t \ge 0$ and $x, y, \bar{x}, \bar{y} \in R^n$,

$|(g_i(\omega, t, x, \bar{x}) - g_i(\omega, t, y, \bar{y}))\, \mathbb{1}_{[\![0, S_k]\!]}(t)| \le a_k(\omega)\,(|x - y| + |\bar{x} - \bar{y}|)$ a.s.

(2) for any $x, y \in R^n$, $g_i(.,.., x, y)$ is a cadlag. adapted process.

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.7.1).

Theorem 1.7.2. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let g_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \times R^n \to R^n$ be random functions, $A : \Omega \times R_+ \to R_+$ be right continuous adapted processes which satisfy the following conditions:

(1) for every $1 \le i \le m, \; t \ge 0$ and $x, y, \bar{x}, \bar{y} \in R^n$,

$|g_i(\omega, t, x, \bar{x}) - g_i(\omega, t, y, \bar{y})| \le A(\omega, t)\,(|x - y| + |\bar{x} - \bar{y}|)$ a.s.

(2) for any $x, y \in R^n$, $1 \le i \le m$, $g_i(.,.., x, y)$ is a cadlag. adapted process.

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.7.1).

Theorem 1.7.3. Let M_i $(1 \le i \le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Let g_i $(1 \le i \le m) : \Omega \times R_+ \times R^n \times R^n \to R^n$ be random functions, $\{a_k\}$ be a sequence of positive real numbers and $\{S_k\}$ be a nondecreasing sequence of stopping times which satisfy the following conditions:

(1) for every $k = 1, 2, \;\;, \; 1 \le i \le m, \; t \ge 0$ and $x, y, \bar{x}, \bar{y} \in R^n$,

$|g_i(\omega, t, x, \bar{x}) - g_i(\omega, t, y, \bar{y})\, \mathbb{1}_{[\![0, S_k]\!]}(t)| \le a_k\,(|x - y| + |\bar{x} - \bar{y}|)$ a.s.

(2) for any $x, y \in R^n$, $g_i(.,.., x, y)$ is a cadlag. adapted process.

Then there exist a stopping time ρ and $X \in \mathcal{O}$ such that

(1) $\lim_{k \to \infty} T_k(X) \wedge S_k = \rho$ a.s.;

(2) for any stopping time σ with $\sigma < \rho$ a.s., we have $X^{\sigma-} \in \mathcal{X}$ and

$$X(\sigma) = H(\sigma) + \sum_{i=1}^{m} \int_0^{\sigma} f_i(\cdot, s, X(s-), X^d(s-))\, dM_i(s) \qquad (1.5.2)$$

Furthermore, if there exist $Y \in \mathcal{O}$ and a stopping time τ such that $X^{\tau-} \in \mathcal{X}$ and

$$Y(t) = H(t) + \sum_{i=1}^{m} \int_0^t f_i(\cdot, s, Y(s-), Y^d(s-))\, dM_i(s), \quad 0 \le t \le \tau \qquad (1.5.3)$$

we have

(3) $\rho' := \lim_{k\to\infty} S_k \wedge \tau \le \rho$ a.s. and $Y^{\rho'-} = X^{\rho'-}$;

(4) $\tau \le \rho$ a.s. on $\left\{\omega : \lim_{k\to\infty} T_k(X) \le \lim_{k\to\infty} S_k\right\}$.

Theorem 1.7.4. Let M_i $(1\le i\le m) \in \mathcal{M}$ and $H \in \mathcal{X}$. Assume

$$E\Big(\sup_{t\in D} |A(t-d(t))|^p\Big)^{1/p} \le C = \text{Const.} < \infty \quad \text{a.s.}$$

Let g_i $(1\le i\le m) : \Omega \times R_+ \times R^n \times R^n \to R^n$ be random functions, $\{a_k(\omega)\}$ and $\{K_k(\omega)\}$ be two sequences of positive random variables, and $\{S_k\}$ be a sequence of stopping times with $S_k \uparrow \infty$ a.s. which satisfy the following conditions:

(1) for every $k = 1, 2, \ldots,$ $1 \le i \le m$, $t \ge 0$ and $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le k$,

$$\left|\big(g_i(\omega, t, x, \bar{x}) - g_i(\omega, t, y, \bar{y})\big) 1_{[\![0, S_k]\!]}(t)\right| \le a_k(\omega)(|x - y| + |\bar{x} - \bar{y}|) \quad \text{a.s.}$$

(2) for every $k = 1, 2, \ldots,$ $1 \le i \le m$, $t \ge 0$ and $x, y \in R^n$,

$$|g_i(\omega, t, x, y) 1_{[\![0, S_k]\!]}(t)| \le K_k(\omega)(1 + |x| + |y|) \quad \text{a.s.}$$

(3) for any $x, y \in R^n$, $1 \le i \le m$, $g_i(\cdot, \cdot, x, y)$ is a cadlag. adapted process.

Then there exists a unique solution in $\mathcal{X}$ to Eq.(1.7.1).

Chapter II

Linear Stochastic Integral Equations

2.1. INTRODUCTION

This chapter presents the explicit representation of the solutions to linear SIES. The classical exponential formula of semimartingales is first introduced in section 2.2. We then, in section 2.3, define the concept of stochastic Wronskian determinant and establish stochastic Liouville's formula. As an application of stochastic Liouville's formula we give the explicit representation of the solutions to linear SIES in section 2.4. In addition, we describe a method to solve constant coefficient linear SIES in section 2.5. Some examples are worked out to illustrate the usefulness of our results. Sections 2.3 - 2.5 are mainly based on my paper Mao [1] published in 1983 (cf. Mathematical Review 85g : 60069).

2.2. EXPONENTIAL FORMULA OF SEMIMARTINGALES

For the convenience, let us introduce the exponential formula of semimartingales which will be used to prove our results in the following sections.

Let $M \in \mathcal{M}$. By Theorem 1.4.4 or Theorem 1.5.2, there exists a unique solution, say $\mathcal{E}(M)$ usually, in $\mathcal{M}$ to the following one-dimensional linear SIES

$$X(t) = 1 + \int_0^t X(s-)\, dM(s), \quad t \geq 0 \tag{2.2.1}$$

The explicit expression of this solution is said to be the exponential formula of semimartingales due to Doléans-Dade [1] as follows:

Theorem 2.2.1. Let $M \in \mathcal{M}$ with $M(0) = 0$. Then the unique solution $\mathcal{E}(M)$ of Eq.(2.2.1) can be expressed as

$$\mathcal{E}(M)(t) = \exp\Big(M(t) - \frac{1}{2}\langle M^c, M^c\rangle(t) \Big) \prod_{0<s\le t} (1 + \Delta M(s))\, e^{-\Delta M(s)}$$

where M^c is the continuous local martingale part of M and $\Delta M(s) = M(s) - M(s-)$.

2.3. STOCHASTIC LIOUVILLE'S FORMULA

Let $\mathcal{M}^c$ denote the family of all one-dimensional continuous semimartingales. Let $M = (M_1, ..., M_n)$, where M_k $(1\le k\le m) \in \mathcal{M}^c$ with $M_k(0) = 0$. If $A = (a_{ij})$ is a matrix, we use A' to define its transposed matrix (a_{ji}). Let $B = (B^1, ..., B^m)$, where $B^k = (b^k_{ij})_{n\times n}$, $1 \le k \le m$, are all nxn matrixes. Throughout this chapter we assume b^k_{ij} are all bounded predictable processes. Let us consider the following n-dimensional linear SIES

$$X(t) = a + \sum_{k=1}^{m} \int_0^t B^k(s)\, X(s)\, dM_k(s), \qquad t \ge 0 \tag{2.3.1}$$

where $X = (X_1, ..., X_n)'$ and $a \in R^n$. By Theorem 1.5.1, there is a unique solution in $\mathcal{X}$ to Eq.(2.3.1).

Let $A = (a_{ij})_{n\times n}$ be an arbitrary matrix. Let $(X_{1k}, ..., X_{nk})'$ denote the unique solution of Eq.(2.3.1) with $a = (a_{1k}, ..., a_{nk})'$. Set

$$\mathcal{E}^A(M, B) = (X_{ij})_{n\times n} \tag{2.3.2}$$

which is said to be the stochastic solution matrix. Particularly, when A is the identity matrix, we use $\mathcal{E}(M, B)$ to stand for $\mathcal{E}^A(M, B)$ simply. It is obvious that $\mathcal{E}^A(M, B) = \mathcal{E}(M, B)\, A$. If the rank of A equals to n, $\mathcal{E}^A(M, B)$ is said to be the stochastic fundamental solution matrix. Clearly, $\mathcal{E}(M, B)$ is a stochastic fundamental solution matrix, which is also called the stochastic standard solution matrix. If $\mathcal{E}^{A_1}(M, B)$ and $\mathcal{E}^{A_2}(M, B)$ are two stochastic fundamental solution matrixes, we have the relation

$$\mathcal{E}^{A_1}(M, B)\, A_1^{-1} = \mathcal{E}^{A_2}(M, B)\, A_2^{-1}$$

which means that stochastic fundamental solution matrixes can express each other.

Denote the determinant of the stochastic solution matrix $\mathcal{E}^A(M, B)$ by

$$W = W^A(M, B) = \det.\ \mathcal{E}^A(M, B) \tag{2.3.3}$$

which is said to be the stochastic Wronskian determinant. Clearly, $W(0) = \det.A$. For the stochastic Wronskian determinant, we have the following theorem.

Theorem 2.3.1. For all $t \geq 0$ we have

$$W(t) = \det.A \cdot \exp\Big\{ \sum_{k=1}^{m} \int_0^t \Big(\sum_{u=1}^{n} b_{uu}^k(s) \Big) dM_k(s)$$

$$- \sum_{k,l=1}^{m} \int_0^t \Big[\frac{1}{2} \sum_{u=1}^{n} b_{uu}^k(s)\, b_{uu}^l(s) + \sum_{1 \leq u < v \leq n} b_{uv}^k(s)\, b_{vu}^l(s) \Big] d\langle M_k^c, M_l^c \rangle(s) \Big\} \tag{2.3.4}$$

which will be called the stochastic Liouville formula.

Proof. It follows from the definition of the determinant that

$$W(t) = \sum_{(l_1, \ldots, l_n)} (-1)^{\tau(l_1, \ldots, l_n)} \prod_{r=1}^{n} X_{rl_r}(t) \tag{2.3.5}$$

Since

$$X_{rl_r}(t) = a_{rl_r} + \sum_{k=1}^{m} \int_0^t \sum_{j=1}^{n} b_{rj}^k(s)\, X_{jl_r}(s)\, dM_k(s)$$

we have, by Itô's formula, that

$$\prod_{r=1}^{n} X_{rl_r}(t) = \prod_{r=1}^{n} a_{rl_r} + \sum_{u=1}^{n} \int_0^t \prod_{r \neq u} X_{rl_r}(s)\, dX_{rl_u}(s)$$

$$+ \sum_{1 \leq u < v \leq n} \int_0^t \prod_{r \neq u,v} X_{rl_r}(s)\, d\langle X_{ul_u}^c, X_{vl_v}^c \rangle(s)$$

$$:= \sum_{j=1}^{3} \nabla_j(l_1, \ldots, l_n) \tag{2.3.6}$$

However

$$\sum_{(l_1, \ldots, l_n)} (-1)^{\tau(l_1, \ldots, l_n)} \nabla_1(l_1, \ldots, l_n) = \det.A \tag{2.3.7}$$

and

$$\sum_{(l_1, \ldots, l_n)} (-1)^{\tau(l_1, \ldots, l_n)} \nabla_2(l_1, \ldots, l_n)$$

$$= \sum_{u=1}^{n} \int_0^t \det \begin{bmatrix} X_{11}(s) & \ldots & X_{1n}(s) \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ dX_{u1}(s) & \ldots & dX_{un}(s) \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ X_{n1}(s) & \ldots & X_{nn}(s) \end{bmatrix}$$

$$= \sum_{u=1}^{n} \sum_{k=1}^{m} \int_0^t b_{uu}^{k}(s)\, W(s)\, dM_k(s) \tag{2.3.8}$$

We also have

$$d\langle X_{ul_u}^{c}, X_{vl_v}^{c}\rangle(s)$$

$$= \sum_{k,l=1}^{m} \Big(\sum_{j=1}^{n} b_{uj}^{k}(s)\, X_{jl_u}(s) \Big) \Big(\sum_{j=1}^{n} b_{vj}^{l}(s)\, X_{jl_v}(s) \Big)\, d\langle M_k^{c}, M_l^{c}\rangle(s)$$

$$= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} b_{ui}^{k}(s)\, b_{vj}^{l}(s)\, X_{jl_u}(s)\, X_{jl_v}(s) \Big)\, d\langle M_k^{c}, M_l^{c}\rangle(s)$$

Noticing

$$\sum_{(l_1, \ldots, l_n)} (-1)^{\tau(l_1, \ldots, l_n)} X_{jl_u}(s)\, X_{jl_v}(s) \prod_{r \neq u,v} X_{rl_r}(s) = \begin{cases} W(s), & \text{if } i = u,\ j = v \\ -W(s), & \text{if } i = v,\ j = u \\ 0, & \text{otherwise} \end{cases}$$

we get

$$\sum_{(l_1,\dots,l_n)} (-1)^{\tau(l_1,\dots,l_n)} \nabla_3(l_1,\dots,l_n) \;=$$

$$= \sum_{1\le u<v\le n} \sum_{k,l=1}^{m} \int_0^t \left(b^k_{uu}(s)\, b^l_{uu}(s) - b^k_{uv}(s)\, b^l_{vu}(s) \right) W(s)\, d\langle M^c_k, M^c_l\rangle(s) \quad (2.3.9)$$

Combining (2.3.5) - (2.3.9), we arrive at

$$W(t) = \det.A + \sum_{u=1}^{n}\sum_{k=1}^{m} \int_0^t b^k_{uu}(s)\, W(s)\, dM_k(s)$$

$$+ \sum_{1\le u<v\le n} \sum_{k,l=1}^{m} \int_0^t \left(b^k_{uu}(s)\, b^l_{uu}(s) - b^k_{uv}(s)\, b^l_{vu}(s) \right) W(s)\, d\langle M^c_k, M^c_l\rangle(s) \quad (2.3.10)$$

If we set

$$Z(t) = \sum_{u=1}^{n}\sum_{k=1}^{m} \int_0^t b^k_{uu}(s)\, dM_k(s)$$

$$+ \sum_{1\le u<v\le n} \sum_{k,l=1}^{m} \int_0^t \left(b^k_{uu}(s)\, b^l_{uu}(s) - b^k_{uv}(s)\, b^l_{vu}(s) \right) d\langle M^c_k, M^c_l\rangle(s) \qquad (2.3.11)$$

then $Z \in \mathcal{M}^c$ with $Z(0) = 0$, and it follows from (2.3.10) that

$$W(t) = \det.A + \int_0^t W(s)\, dZ(s) \qquad (2.3.12)$$

In view of Theorem 2.2.1, we have

$$W(t) = \det.A \cdot \exp\left\{ Z(t) - \frac{1}{2} \langle Z^c, Z^c\rangle(s) \right\} \qquad (2.3.13)$$

However

$$\langle Z^c, Z^c\rangle$$

$$= \left\langle \sum_{u=1}^{n}\sum_{k=1}^{m} b^k_{uu} \cdot M^c_k,\; \sum_{u=1}^{n}\sum_{k=1}^{m} b^k_{uu} \cdot M^c_k \right\rangle$$

$$= \sum_{k,l=1}^{m} \sum_{u,v=1}^{n} b_{uu}^{k} b_{vv}^{k} . \langle M_k^c, M_l^c \rangle$$

$$= \sum_{k,l=1}^{m} \Big\{ \sum_{u=1}^{n} b_{uu}^{k} b_{vv}^{k} . \langle M_k^c, M_l^c \rangle + 2 \sum_{1 \le u < v \le n} b_{uu}^{k} b_{vv}^{k} . \langle M_k^c, M_l^c \rangle \Big\} \quad (2.3.14)$$

Putting this and (2.3.12) into (2.3.13) we deduce desired formula (2.3.4) which completes the proof.

We now consider an important and typical case. Let $M_1 = (M_1(t)) = (t)$ and $(M_2, ..., M_m) = (W_2, ..., W_m)$ an m-1-dimensional Wiener process. Then Eq.(2.3.1) can be expressed as

$$X(t) = a + \int_0^t B^1(s)\, X(s)\, ds + \sum_{k=2}^{m} \int_0^t B^k(s)\, X(s)\, dW_k(s), \qquad t \ge 0$$

which is equivalent to the n-dimensional Itô's differential equation

$$dX(t) = B^1(t)\, X(t)\, dt + \sum_{k=2}^{m} B^k(t)\, X(t)\, dW_k(t) \;\; (t \ge 0), \;\; X(0) = a \qquad (2.3.15)$$

We obtain from Theorem 2.3.1 the following useful

Corollary 2.3.2. For the stochastic Wronskian determinant W of the stochastic solution matrix to Eq.(2.3.15) we have

$$W(t) = \det.A \cdot \exp\Big\{ \int_0^t \Big(\sum_{u=1}^{n} b_{uu}^{k}(s) \Big) ds -$$

$$- \sum_{k=2}^{m} \int_0^t \Big[\frac{1}{2} \sum_{u=1}^{n} (b_{uu}^{k}(s))^2 + \sum_{1 \le u < v \le n} b_{uv}^{k}(s)\, b_{vu}^{k}(s) \Big] ds \Big\} \qquad (2.3.16)$$

for all $t \ge 0$.

2.4 EXPLICIT EXPRESSION OF SOLUTIONS

Let us now consider the following linear SIES

$$X(t) = H(t) + \sum_{k=1}^{m} \int_0^t B^k(s)\, X(s-)\, dM_k(s), \quad t \geq 0 \tag{2.4.1}$$

where $H = (H_1, ..., H_n)' \in \mathcal{M}_n$, i.e., an n-dimensional semimartingale (may not be continuous). By Theorem 1.4.4, there exists a unique solution in $\mathcal{M}_n$ to Eq.(2.4.1). Denote by $\mathcal{E}_H(M, B)$ this semimartingale solution.

In this section we will describe the relation between $\mathcal{E}_H(M, B)$ and the stochastic fundamental solution matrix $\mathcal{E}^A(M, B)$ of the corresponding Eq.(2.3.1), which means that the solution $\mathcal{E}_H(M, B)$ can be explicitly expressed provided we get a stochastic fundamental solution matrix $\mathcal{E}^A(M, B)$ to Eq.(2.3.1).

Since the stochastic fundamental solution matrixes of Eq.(2.3.1) can express each other, we only need to give the relation between $\mathcal{E}_H(M, B)$ and $\mathcal{E}(M, B)$ which is described by the following theorem.

Theorem 2.4.1. The solution of Eq.(2.4.1) can be explicitly expressed as

$$\mathcal{E}_H(M, B)(t) = \mathcal{E}(M, B)(t) \Big\{ H(0) + \int_0^t \mathcal{E}(M, B)^{-1}(s)\, dH(s)$$

$$- \sum_{k=1}^{m} \int_0^t \mathcal{E}(M, B)^{-1}(s)\, B^k(s)\, d\langle M_k^c, H^c\rangle(s) \Big\} \tag{2.4.2}$$

for all $t \geq 0$, where $\mathcal{E}(M, B)$ is the stochastic standard solution matrix of the corresponding Eq.(2.3.1) and

$$\langle M_k^c, H^c\rangle = (\langle M_k^c, H_1^c\rangle, ..., \langle M_k^c, H_n^c\rangle)'$$

Proof. Let $\mathcal{E}(M, B) = Z = (Z_{ij})_{n\times n}$. Set

$$\mathcal{E}_H(M, B) = X = Z\,C \tag{2.4.3}$$

where $C = (C_1, ..., C_n)'$ is an n-dimensional semimartingale. Clearly, (2.4.3) can expressed as follows:

$$X_i = \sum_{j=1}^{n} Z_{ij} C_j, \quad i = 1, 2, \ldots, n$$

Using the integral by parts (cf. Yan [1], Theorem 10.5, p.251), we have

$$dX_i = \sum_{j=1}^{n} (Z_{ij}\, dC_j + C_j^- \, dZ_{ij} + d[Z_{ij}, C_j])$$

where $C_j^- = (C_j(t-))$. Since

$$dZ_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{n} b_{il}^{k} Z_{lj}\, dM_k$$

we get

$$dX_i = \sum_{j=1}^{n} Z_{ij}\, dC_j + \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{j=1}^{n} b_{il}^{k} Z_{lj} C_j^- \, dM_k$$

$$+ \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{j=1}^{n} b_{il}^{k} Z_{lj}\, d[M_k, C_j]$$

which is

$$dX = Z\, dC + \sum_{k=1}^{m} B^k X^- \, dM_k + \sum_{k=1}^{m} B^k Z\, d[M_k, C] \qquad (2.4.4)$$

where $X^- = (X(t-))$ and $[M_k, C] = ([M_k, C_1], \ldots, [M_k, C_1])'$. However, we have from (2.4.1) that

$$dX = dH + \sum_{k=1}^{m} B^k X^- \, dM_k$$

Consequently

$$Z\, dC + \sum_{k=1}^{m} B^k Z\, d[M_k, C] = dH \qquad (2.4.5)$$

In view of Theorem 2.3.1, the inverse matrix Z^{-1} of Z is almost everywhere well defined. It is also clear that every element of Z^{-1} is a continuous predictable process, hence, a local bounded predictable process. Therefore, we have from (2.4.5) that

$$dC = Z^{-1}\, dH + \sum_{k=1}^{m} Z^{-1} B^k Z\, d[M_k, C] \qquad (2.4.6)$$

which implies

$$dC^c = Z^{-1}\, dH^c$$

where $C^c = (C_1^c, \ldots, C_n^c)'$ and $H^c = (H_1^c, \ldots, H_n^c)'$. Recalling M_k is a continuous semimartingale, we get

$$d[M_k, C] = d\langle M_k^c, C^c\rangle = Z^{-1}\, d\langle M_k^c, H^c\rangle$$

Putting this into (2.4.6) we obtain

$$dC = Z^{-1}\, dH + \sum_{k=1}^{m} Z^{-1} B^k\, d\langle M_k^c, H^c\rangle \tag{2.4.7}$$

namely, since $C(0) = H(0)$,

$$C(t) = H(0) + \int_0^t Z^{-1}(s)\, dH(s) + \sum_{k=1}^{m} \int_0^t Z^{-1}(s)\, B^k(s)\, d\langle M_k^c, H^c\rangle(s)$$

The desired formula (2.4.2) follows by putting this into (2.4.3) and the proof is complete.

We now let $(W_2, \ldots, W_m)$ be defined as in section 2.3 and

$$H(t) = H(0) + \int_0^t \alpha(s)\, ds + \sum_{k=2}^{m} \int_0^t \beta^k(s)\, dW_k(s), \quad t \geq 0$$

where $H(0)$ is an n-dimensional $\mathcal{F}_0$-measurable random vector, $\alpha = (\alpha_1, \ldots, \alpha_n)'$, $\beta^k = (\beta_1^k, \ldots, \beta_n^k)'$ and α_i, β_i^k are all bounded predictable processes. In this case, Eq.(2.4.1) can be expressed as

$$X(t) = H(0) + \int_0^t (\alpha(s) + B^1(s))\, X(s)\, ds$$

$$+ \sum_{k=2}^{m} \int_0^t (\beta^k(s) + B^k(s))\, X(s)\, dW_k(s), \quad t \geq 0 \tag{2.4.8}$$

which is Itô's differential equation

$$dX(t) = (\alpha(s) + B^1(s))\, X(s)\, ds + \sum_{k=2}^{m} (\beta^k(s) + B^k(s))\, X(s)\, dW_k(s), \quad t \geq 0$$

$$X(0) = H(0) \tag{2.4.9}$$

By Theorem 2.4.1 we have the following

Corollary 2.4.2. The unique solution, say $\mathcal{E}_H(M, B)$, of Eq.(2.4.9) can be expressed as

$$\begin{aligned} \mathcal{E}_H(M, B)(t) = \mathcal{E}(M, B)(t) \Big\{ H(0) &+ \int_0^t \mathcal{E}(M, B)^{-1}(s)\, \alpha(s)\, ds \\ &+ \sum_{k=2}^{m} \int_0^t \mathcal{E}(M, B)^{-1}(s)\, \beta^k(s)\, dW_k(s) \\ &- \sum_{k=2}^{m} \int_0^t \mathcal{E}(M, B)^{-1}(s)\, B^k(s)\, \beta^k(s) ds \Big\} \end{aligned} \tag{2.4.10}$$

for all $t \geq 0$, where $\mathcal{E}(M, B)$ is the stochastic standard solution matrix of the corresponding Eq.(2.3.15).

Example 2.4.3. The following equation (the well-known Langevin's equation) can be formally derived from Newton's second law

$$v(t) = v_0 + F(t) - \frac{\lambda}{m} \int_0^t v(s)\, ds, \quad t \geq 0 \quad (m, \lambda > 0) \tag{2.4.11}$$

where v_0 is the initial force acting on the particle, λ is the coefficient of friction figuring in Stoke's law and F means the force acting on the particle by the random collisions and may be regarded as a continuous martingale with initial condition $F(0) = 0$. By Theorem 2.4.1, the solution of this equation has the form

$$v(t) = v_0 e^{-\lambda t/m} + \int_0^t e^{-\lambda(t-s)/m}\, dF(s), \qquad t \geq 0 \tag{2.4.12}$$

Hence, we can easily deduce the mean

$$E\,v(t) = v_0\, e^{-\lambda t/m}$$

and the variance

$$E(\,v(t) - E\,v(t)\,)^2 = \int_0^t e^{-2\lambda(t-s)/m}\, d\langle F, F\rangle(s)$$

for all $t \geq 0$.

Example 2.4.4. The analogous electrical problem is an L - R circuit described for the current I(t) by the equation

$$L\,dI(t) + R\,I(t-)\,dt = de(t) \qquad (2.4.13)$$

where R is the resistance (in ohms), L is the inductance (in henries) and e(t) is a purely fluctuating electromotive force (a thermal noise source) and can be regarded as a semimartingale with $e(0) = 0$. In the case of a capacitance C (in farads) the equation describing the oscillating electrical circuit with thermal noise is

$$L\,d\dot{I}(t) + \{R\,\dot{I}(t-) + \frac{1}{C} I(t-)\}\,dt = de(t) \qquad (2.4.14)$$

Let $J = (J_1, J_2)' = (\dot{I}, I)'$. Then Eq.(2.4.14) is equivalent to

$$dJ(t) = B\,J(t-)\,dt + \beta\,de(t) \qquad (2.4.15)$$

where

$$B = \begin{bmatrix} -\frac{1}{LC} & -\frac{R}{L} \\ 1 & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By Theorem 2.4.1, we have

$$J(t) = e^{Bt} J(0) + \int_0^t e^{B(t-s)}\, \beta\, de(s) \qquad (2.4.16)$$

If we let $L = 0.5$, $C = 0.5$ and $R = 1.5$, we have

$$e^{Bt} = \frac{1}{2}\begin{bmatrix} 3\,e^{-3t} - e^{-t}, & -e^{-3t} + e^{-t} \\ 3\,e^{-3t} - 3\,e^{-t}, & -e^{-3t} + 3\,e^{-t} \end{bmatrix}$$

Therefore

$$I(t) = J_2(t) = \frac{1}{2}\left[(\,3\dot{I}(0) - I(0)\,)\,e^{-3t} - 3\,(\,\dot{I}(0) - I(0)\,)\,e^{-t}\right]$$

$$+\ \frac{3}{2}\int_0^t (e^{-3(t-s)} - e^{-(t-s)}) \, de(s) \tag{2.4.17}$$

Example 2.4.5. A very interesting economic problem is the exchange rate dynamic model, i. e., the stochastic Dornbusch model (cf. Dornbusch [1]). It describes the price index and the exchange rate by the equation

$$\begin{aligned} dX(t) &= (\alpha(t) + A\,X(t))\, dt + \sigma\, dW(t), \qquad t \geq 0 \\ X(0) &= x_o, \quad \text{a.s.} \end{aligned} \tag{2.4.18}$$

where $X = (X_1, X_2)'$, X_1 and X_2 represent the price index (in logs) of domestic final product and the exchang rate (in logs) defined as the foreign currency price of domestic currency respectively, $\alpha : R_+ \to R^2$ is bounded and Borel measurable, A is an 2×2 constant matrix, $\sigma = (\sigma_1, 0)'$, $x_o = (x_{o1}, x_{o2})' \in R^2$ and W is an one-dimensional standard Wiener process. By Corollary 2.4.2, the solution of Eq.(2.4.18) has the form

$$X(t) = e^{At} x_o + \int_0^t e^{A(t-s)} \alpha(s)\, ds + \int_0^t e^{A(t-s)} \sigma\, dW(s), \quad t \geq 0 \tag{2.4.19}$$

which deduces that X is a two-dimensional Gaussian process. We also have from (2.4.19) that

$$E\,X(t) = e^{At} x_o + \int_0^t e^{A(t-s)} \alpha(s)\, ds$$

and

$$B(t, \tau) := E\left\{ (X(t) - E\,X(t)) (X(t+\tau) - E\,X(t+\tau))' \right\} =$$

$$= E\left\{ \int_0^t e^{A(t-s)} \sigma\, dW(s) \int_0^{t+\tau} \sigma' e^{A'(t+\tau-s)}\, dW(s) \right\}$$

$$= \left(\int_0^t e^{A(t-s)} \sigma \sigma' e^{A'(t-s)}\, ds \right) e^{A'\tau} = B(t, 0)\, e^{A'\tau}, \quad t, \tau \geq 0 \tag{2.4.20}$$

We now assume

$$A = \begin{bmatrix} 1, & -2 \\ 3, & -4 \end{bmatrix}$$

and we have

$$e^{At} = \begin{bmatrix} 3e^{-t} - 2e^{-2t}, & -6e^{-t} + 6e^{-2t} \\ e^{-t} - e^{-2t}, & -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

Hence

$$B(t,0) = \sigma_1^2 \begin{bmatrix} 1.5 - 4.5e^{-2t} + 4e^{-3t} - e^{-4t}, & \frac{1}{3} - 1.5e^{-2t} + \frac{5}{3}e^{-3t} - 0.5e^{-4t} \\ \frac{1}{3} - 1.5e^{-2t} + \frac{5}{3}e^{-3t} - 0.5e^{-4t}, & \frac{1}{12} - \frac{1}{2}e^{-2t} + \frac{2}{3}e^{-3t} - \frac{1}{4}e^{-4t} \end{bmatrix}$$

Furthermore, if we let $\alpha(t) = (\sin(t), \cos(t))'$, we can get

$$\begin{aligned} E\,X_1(t) &= e^{-t}(3x_{01} - 6x_{02} - 3) \\ &+ e^{-2t}(-2x_{01} + 6x_{02} - 9\sin(t) + 3\cos(t) + 2.5) \\ &+ 2.5e^{-4t}(\sin(t) - \cos(t)) \end{aligned}$$

and

$$\begin{aligned} E\,X_2(t) &= e^{-t}(x_{01} - 2x_{02} - 1) \\ &+ e^{-2t}(-x_{01} + 3x_{02} - 3\sin(t) + \cos(t) - 1.25) \\ &+ 1.25e^{-4t}(\sin(t) - \cos(t)) \end{aligned}$$

By the property of the normal distribution, we also have

$$P\{\omega : |X_1(t) - E\,X_1(t)| \le 3\sigma_1(1.5 - 4.5e^{-2t} + 4e^{-3t} - e^{-4t})^{1/2}\} = 99.98\%$$

and

$$P\{\omega : |X_2(t) - E\,X_2(t)| \le 3\sigma_1(\frac{1}{12} - \frac{1}{2}e^{-2t} + \frac{2}{3}e^{-3t} - \frac{1}{4}e^{-4t})^{1/2}\} = 99.98\%$$

2.5. CONSTANT COEFFICIENT LINEAR SIES

Let M be a continuous semimartingale with $M(0) = 0$ and H an n-dimensional semimartingale, D an n×n constant matrix. Our aim is to describe a method to solve the constant coefficient linear SIES

$$X(t) = H(t) + \int_0^t D\, X(s-)\, dM(s), \quad t \geq 0 \tag{2.5.1}$$

In view of Theorem 2.4.1, we only need to discuss how to solve the following equation

$$X(t) = a + \int_0^t D\, X(s-)\, dM(s), \quad t \geq 0 \tag{2.5.2}$$

where $a \in R^n$.

It is well known that for any nxn constant matrix D there exists an invertible matrix P such that $P^{-1} D P$ is the Jordan canonical matrix, namely,

$$P^{-1} D P = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_p \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda_i & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix}_{k_i \times k_i}$$

and λ_i $(1 \leq i \leq p)$ are the characteristic roots of D, some of which may be thesame. Set

$$Z = P^{-1} X, \qquad \alpha = P^{-1} a \tag{2.5.3}$$

then Z is the solution of the equation

$$Z(t) = \alpha + \int_0^t P^{-1} D\, P\, Z(s-)\, dM(s), \quad t \geq 0 \tag{2.5.4}$$

which is p-independent multi-dimensional linear SIES as follows:

$$\begin{cases} Z_1(t) = \alpha_1 + \int_0^t (\lambda_1 Z_1(s-) + Z_2(s-))\, dM(s) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_{k_1}(t) = \alpha_{k_1} + \int_0^t \lambda_1 Z_{k_1}(s-)\, dM(s) \end{cases} \tag{2.5.5}$$

....

$$\begin{cases} Z_{n-k_p+1}(t) = \alpha_{n-k_p+1} + \int_0^t (\lambda_p Z_{n-k_p+1}(s-) + Z_{n-k_p+2}(s)\, dM(s) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Z_n(t) = \alpha_n + \int_0^t \lambda_p Z_n(s-)\, dM(s) \end{cases} \tag{2.5.6}$$

We can now solve these equations independently. For example, for Eq.(2.5.5), it follows from the last equation in (2.5.6) by using Theorem 2.2.1 that

$$Z_{k_1}(t) = \alpha_{k_1} \exp\left\{ \lambda_1 M(t) - \frac{\lambda_1^2}{2} \langle M^c, M^c \rangle(t) \right\} := \alpha_{k_1}\, \mathcal{E}(\lambda_1 M)(t) \tag{2.5.7}$$

Putting this into the last second equation in (2.5.5) we get

$$Z_{k_1-1}(t) = \alpha_{k_1-1} + \int_0^t \alpha_{k_1}(s)\, \mathcal{E}(\lambda_1 M)(s)\, dM(s) + \int_0^t \lambda_1 Z_{k_1-1}(s-)\, dM(s)$$

By Theorem 2.4.1, we have

$$Z_{k_1-1}(t) = \mathcal{E}(\lambda_1 M)(t) \left\{ \alpha_{k_1-1} + \alpha_{k_1} (M(t) - \lambda_1 \langle M^c, M^c \rangle(t)) \right\}$$

Similarly, we can get

$$Z_{k_1-2}(t) = \mathcal{E}(\lambda_1 M)(t) \sum_{i=1}^{2} \alpha_{k_1-2+i}\, I(M, \lambda_1, i)(t)$$

....

$$Z_1(t) = \mathcal{E}(\lambda_1 M)(t) \sum_{i=1}^{k_1-1} \alpha_{1+i}\, I(M, \lambda_1, i)(t)$$

where

$$I(M, \lambda, k)(t) = \begin{cases} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} d(M - \lambda\langle M^c, M^c\rangle)(s_k) \ldots d(M - \lambda\langle M^c, M^c\rangle)(s_1), & k \geq 1 \\ 1, \quad k = 0 \end{cases}$$

We can similarly solve other equations. Thus we get the unique solution of Eq.(2.5.4) as follows:

$$Z_1(t) = \mathcal{E}(\lambda_1 M)(t) \sum_{i=1}^{k_1-1} \alpha_{1+i} I(M, \lambda_1, i)(t)$$

....

$$Z_{k_1}(t) = \mathcal{E}(\lambda_1 M)(t)\, \alpha_{k_1}$$

$$Z_{k_1+1}(t) = \mathcal{E}(\lambda_2 M)(t) \sum_{i=1}^{k_2-1} \alpha_{k_1+1+i} I(M, \lambda_2, i)(t)$$

....

$$Z_n(t) = \mathcal{E}(\lambda_p M)(t)\, \alpha_n$$

We then get the solution $X = P\,Z$ to Eq.(2.5.2).

Howerer, generally speaking, it is not easy to find matrix P so that the above method is not convenient in applications. Fortunately, we can analyse this method in more details in order to obtain another more convenient method.

In fact, let λ_i ($1 \leq i \leq p$) be the characteristic roots of D and k_i the number of the repeated roots λ_i. It is well known that the number of Jordan decomposition matrixes corresponding to λ_i may be more than one, but the order of every Jordan decomposition matrix should not be bigger than k_i. Hence, it follows from the above discussion that the solution of Eq.(2.5.2) can be written as

$$X = C\,\mathbb{H} \tag{2.5.7}$$

where $C = (C_{ij})_{n \times n}$ is a constant matrix to be defined and

$$\mathbb{H} = \begin{bmatrix} H(M, \lambda_1, 0) \\ \cdots\cdots\cdots \\ H(M, \lambda_1, k_1-1) \\ H(M, \lambda_2, 0) \\ \cdots\cdots\cdots \\ H(M, \lambda_p, k_p-1) \end{bmatrix}$$

in which

$$H(M, \lambda, k) = I(M, \lambda, k)\, \mathcal{E}(\lambda M), \quad k = 0, 1, \ldots$$

It is easy to show that $H(M, \lambda, k)$ ($k \geq 1$) is the unique semimartingale solution of the equation

$$Y(t) = \int_0^t H(M, \lambda, k-1)(s)\, dM(s) + \int_0^t \lambda\, Y(s)\, dM(s) \qquad (\lambda \neq 0) \tag{2.5.8}$$

and $H(M, \lambda, 0) = \mathcal{E}(\lambda M)$ satisfies the following equation

$$Y(t) = 1 + \int_0^t \lambda\, Y(s)\, dM(s) \tag{2.5.9}$$

Therefore, we deduce

$$\int_0^t H(M, \lambda, k)(s)\, dM(s) = \sum_{i=1}^{k} \frac{(-1)^i}{\lambda^{i+1}} H(M, \lambda, k-i) + \frac{(-1)^{k+1}}{\lambda^{k+1}} \tag{2.5.10}$$

for all $k \geq 0$ and $\lambda \neq 0$. When $\lambda = 0$, we have

$$\int_0^t H(M, 0, k)(s)\, dM(s) = H(M, \lambda, k+1)(s), \quad k \geq 0 \tag{2.5.11}$$

In order to determine the constant matrix C we put (2.5.7) into (2.5.2) and then use (2.5.10) and (2.5.11) to calculate the integrals. Comparing the corresponding coefficients of each $H(M, \lambda_i, k)$ we get a linear equation for C_{ij}. Solving this equation and putting them into (2.5.7) we finally obtain the solution of Eq.(2.5.2). We now give an example to illustrate this method.

Example 2.5.1. Consider the equation

$$X(t) = a + \int_0^t D\,X(s-)\,dM(s), \quad t \geq 0 \tag{2.5.12}$$

where

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic roots of D are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 1$. Thus

$$\mathbb{H} = \begin{bmatrix} 1 \\ H(M, 1, 0) \\ H(M, 1, 1) \end{bmatrix} \tag{2.5.13}$$

Substituting C $\mathbb{H}$ for X in Eq.(2.5.12) and applying (2.510) and(2.5.11) we get

$$\begin{aligned} & C_{11} + C_{12}\,H(M, 1, 0)(t) + C_{13}\,M(M, 1, 1) \\ = \; & 1 + (C_{21} + C_{31})\,M(t) + (C_{22} + C_{32})\,(H(M, 1, 0)(t) - 1) \\ + \; & (C_{23} + C_{33})\,(H(M, 1, 1)(t) - H(M, 1, 0)(t) + 1) \end{aligned}$$

$$\begin{aligned} & C_{21} + C_{22}\,H(M, 1, 0)(t) + C_{23}\,M(M, 1, 1) \\ = \; & (C_{11} + C_{21} - C_{31})\,M(t) + (C_{12} + C_{22} - C_{32})\,(H(M, 1, 0)(t) - 1) \\ + \; & (C_{13} + C_{23} - C_{33})\,(H(M, 1, 1)(t) - H(M, 1, 0)(t) + 1) \end{aligned}$$

$$\begin{aligned} & C_{31} + C_{32}\,H(M, 1, 0)(t) + C_{33}\,M(M, 1, 1) \\ = \; & (C_{21} + C_{31})\,M(t) + (C_{22} + C_{32})\,(H(M, 1, 0)(t) - 1) \\ + \; & (C_{23} + C_{33})\,(H(M, 1, 1)(t) - H(M, 1, 0)(t) + 1) \end{aligned}$$

Comparing the coefficients we get the following linear equation

$$\begin{aligned} C_{11} &= 1 + C_{23} + C_{33} - C_{22} - C_{32} \\ 0 &= C_{21} + C_{31} \\ C_{12} &= C_{22} + C_{32} \\ C_{13} &= C_{23} + C_{33} \end{aligned}$$

$$
\begin{aligned}
C_{21} &= -C_{12} - C_{22} + C_{32} + C_{13} + C_{23} - C_{33} \\
0 &= C_{11} + C_{21} - C_{31} \\
C_{22} &= C_{12} + C_{22} - C_{32} - C_{13} - C_{23} + C_{33} \\
C_{23} &= C_{13} + C_{23} - C_{33} \\
C_{31} &= -C_{22} - C_{32} + C_{23} + C_{33} \\
C_{32} &= C_{22} + C_{32} - C_{23} + C_{33} \\
C_{33} &= C_{23} + C_{33}
\end{aligned}
$$

whose solution is

$$
C = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
$$

Therefore, we finally get the solution of Eq.(2.5.13) as follows:

$$X_1(t) = 2 + \left(-1 + M(t) - \langle M^c, M^c\rangle(t)\right) \exp\left(M(t) - \frac{1}{2}\langle M^c, M^c\rangle(t)\right)$$

$$X_2(t) = -1 + \exp\left(M(t) - \frac{1}{2}\langle M^c, M^c\rangle(t)\right)$$

$$X_3(t) = 1 + \left(-1 + M(t) - \langle M^c, M^c\rangle(t)\right) \exp\left(M(t) - \frac{1}{2}\langle M^c, M^c\rangle(t)\right)$$

Chapter III

Stochastic Stability and Boundedness

3.1. INTRODUCTION

In 1959, Bertram and Sarachik [1] applied the stochastic Lyapunov function to stochastic differential equations. From that time on, the study of stochastic stability was further emphasized by other mathematicians such as Bucy [1] studied more systematically the concept of the stochastic Lyapunov function to stochastic differential equations in 1965. During the same period, Khasminskii [1], Kushner [1], Friedman [1] and others extended the well-known Lyapunov theorem to Itô's stochastic differential equations. We also published several papers on this field during past few years (cf. Li and Mao [1-3]).

This chapter is devoted to new results on the stochastic stability and boundedness. We will pay our attention to stochastic integral equations with respect to semimartingales rather than classical Itô's differential equations with respect to Wiener processes. We first give in section 3.2 some new Lebesgue-Stieltjes integral inequalities of Gronwall-Bellman-Bihari type which are only a little part of my results on this field (cf. Mao [6]). We then establish systematically new results on stochastic stability and boundedness in section 3.3 - 3.6. The results in section 3.2 and 3.5 were published in Quarterly Journal of Mathematics (Oxford) 40 (3) 1989 (cf. Mao [4]).

3.2. INEQUALITIES OF GRONWALL-BELLMAN-BIHARI TYPE

In this section we will give some Lebesgue-Stieltjes integral inequalities which are the natural extensions of the Gronwall-Bellman-Bihari type inequalities. They will be used to prove our results in the following sections.

Let $T > 0$ and $\mu : [0, T] \to R_+$ be a continuous nondecreasing function with $\mu(0) = 0$. Define

$$\sigma(t) = \inf\{s : s \in [0, T], \ \mu(s) > t\} \wedge T, \quad 0 \le t \le \mu(T) \quad (3.2.1)$$

$$\sigma_-(t) = \inf\{s : s \in [0, T], \ \mu(s) \ge t\}, \quad 0 \le t \le \mu(T) \quad (3.2.2)$$

We have following relations (cf. Yan [1], p.26)

$$\mu(\sigma(t)) = \mu(\sigma_-(t)) = t, \quad 0 \le t \le \mu(T)$$

$$\sigma_-(t) \le \sigma(t), \quad 0 \le t \le \mu(T)$$

$$\sigma_-(\mu(t)) \le t \le \sigma(\mu(t)), \quad 0 \le t \le T$$

For the convenience we now state

Lemma 3.2.1 (Lebesgue's lemma. cf. Yan [1], Lemma 1.43). If $f(\cdot)$ is an either bounded or nonnegative Borel function defined on $[0, T]$, then

$$\int_0^t f(s)\, d\mu(s) = \int_0^{\mu(t)} f(\sigma_-(s))\, ds = \int_0^{\mu(t)} f(\sigma(s))\, ds$$

holds for all $t \in [0, T]$.

The following lemma is a generalization of the well-known Gronwall-Bellman inequality (cf. Bellman and Cooke [1]) which is available for Lebesgue-Stieltjes integrals.

Lemma 3.2.2. Let $T > 0$ and $\mu : [0, T] \to R_+$ be a continuous nondecreasing function with $\mu(0) = 0$. Let $u, K : [0, T] \to R_+$ be integrable functions with respect to μ. Let $u_0 \ge 0$ be a constant. Then, the inequality

$$u(t) \le u_0 + \int_0^t K(s)\, d\mu(s) + \int_0^t u(s)\, d\mu(s), \quad 0 \le t \le T \quad (3.2.3)$$

implies

$$u(t) \;\leq\; u_0 e^{\mu(t)} + \int_0^t e^{\mu(t)-\mu(s)} K(s)\, d\mu(s), \qquad 0 \leq t \leq T \tag{3.2.4}$$

Proof. Set

$$w(t) \;=\; u_0 + \int_0^t K(s)\, d\mu(s) \;+\; \int_0^t u(s)\, d\mu(s), \quad 0 \leq t \leq T$$

which is nondecreasing. It is obvious that

$$u(t) \leq w(t), \quad 0 \leq t \leq T$$

and

$$w(t) \;\leq\; u_0 + \int_0^t K(s)\, d\mu(s) \;+\; \int_0^t w(s)\, d\mu(s), \quad 0 \leq t \leq T$$

Hence, for each $v \in [0, \mu(T)]$, we have

$$w(\sigma(v)) \;\leq\; u_0 + \int_0^{\sigma(v)} K(s)\, d\mu(s) \;+\; \int_0^{\sigma(v)} w(s)\, d\mu(s)$$

$$\leq\; u_0 + \int_0^{v} K(\sigma(s)\, ds \;+\; \int_0^{v} w(\sigma(s))\, ds$$

where Lebesgue's Lemma has been used, $\sigma(\cdot)$ and following $\sigma_{-}(\cdot)$ are defined by (3.2.1) and (3.2.2) respectively. In view of Lemma 2.1 of Raghavendra and Ras [1], we deduce that

$$u(\sigma(v)) \;\leq\; w(\sigma(v)) \;\leq\; u_0 e^{v} + \int_0^{v} e^{v-s} K(\sigma(s))\, ds, \qquad 0 \leq t \leq \mu(T)$$

Now, for every $t \in [0, T]$, let $v = \mu(t)$. Thus

$$\sigma_{-}(v) \leq t \leq \sigma(v)$$

and

$$u(t) \;\leq\; w(t) \;\leq\; w(\sigma(v)) \;\leq$$

$$\leq u_0 e^v + \int_0^v e^{v-s} K(\sigma(s))\, ds$$

$$\leq u_0 e^{\mu(t)} + \int_0^{\mu(t)} e^{\mu(t)-\mu(\sigma(s))} K(\sigma(s))\, ds$$

$$\leq u_0 e^{\mu(t)} + \int_0^t e^{\mu(t)-\mu(s)} K(s)\, d\mu(s)$$

which is desired result (2.4). The proof is complete.

The following lemma gives a generalization of Bihari's inequality (cf. Bihari [1]).

Lemma 3.2.3. Let $T > 0$ and $\mu : [0, T] \to R_+$ be a continuous nondecreasing function with $\mu(0) = 0$. Let $u : [0, T] \to R_+$ be a bounded Borel measurable functions. Let $g, f : \mathbb{R}_+ \to \mathbb{R}_+ (= [0, \infty])$ be continuous functions with f strictly increasing, g nondecreasing and $g(v) > 0$ for $v > 0$. Let $u_0 > 0$ be a constant. Assume the inequality

$$f(u(t)) \leq u_0 + \int_0^t g(u(s))\, d\mu(s) \tag{3.2.5}$$

holds for all $0 \leq t \leq T$. Then the inequality

$$u(t) \leq f^{-1}\big(Q^{-1}(Q(u_0) + \mu(t))\big) \tag{3.2.6}$$

remains valid for all $0 \leq t \leq T^*$, where

$$Q(v) = \int_\varepsilon^v \frac{ds}{g(f^{-1}(s))}, \qquad \varepsilon > 0,\ v > 0 \tag{3.2.7}$$

f^{-1} and Q^{-1} denote the inverse mappings of f and Ω respectively,

$$T^* = \max\{t : Q(u_0) + \mu(t) \leq Q(f(\infty)),\ 0 \leq t \leq T\}$$

Proof. For all $t \in [0, T^*]$, define

$$w(t) = u_o + \int_0^t g(u(s))\, d\mu(s)$$

Then

$$f(u(t)) \leq w(t) \qquad \text{and} \qquad u(t) \leq f^{-1}(w(t))$$

By the fundamental theorem of calculus, we have

$$Q(w(t)) - Q(u_o) = \int_0^t \frac{1}{g(f^{-1}(w(s)))}\, g(u(s))\, d\mu(s)$$

$$\leq \int_0^t d\mu(s) = \mu(t)$$

which is (3.2.6) with a change of dummy variable. The proof has been comple

Lemma 3.2.4. Let u(.) and v(.) be non-negative integrable functions defir on [0, T]. Then the inequality

$$u(t) - u(s) \leq -\int_s^t v(\tau)\, u(\tau)\, d\tau, \qquad \text{for all } 0 \leq s < t \leq T \qquad (3.$$

impliesthat

$$u(t) \leq u(0) \exp\left\{-\int_0^t v(\tau)\, d\tau\right\}, \qquad 0 \leq t \leq T \qquad (3.2$$

Proof. Suppose (3.2.9) is false. Then there exists a t_o, $0 < t_o \leq T$, such that

$$u(t_o) > u(0) \exp\left\{-\int_0^{t_o} v(\tau)\, d\tau\right\} \qquad (3.2 \quad)$$

Define

$$t_1 = \sup\left\{t \leq t_o :\ u(t) \leq u(0) \exp\left(-\int_0^t v(\tau)\, d\tau\right)\right\}$$

Clearly, we have $0 \le t_1 < t_o$. It follows from (3.2.8) that u(t) is non-increasir
Hence, we deduce

$$u(t_1) \le \lim_{t \uparrow t_1} u(t) \le u(0) \exp\left\{ - \int_0^{t_1} v(\tau)\, d\tau \right\} \tag{3.2.11}$$

On the other hand, we have

$$u(t) > u(0) \exp\left\{ - \int_0^{t_1} v(\tau)\, d\tau \right\}, \qquad t_1 < t \le t_o$$

Consequently, we yield by (3.2.8) that

$$u(t_o) - u(t_1) < - \int_{t_1}^{t_o} v(s)\, u(0) \exp\left\{ - \int_0^{s} v(\tau)\, d\tau \right\} ds$$

$$= u(0) \exp\left\{ - \int_0^{t_o} v(\tau)\, d\tau \right\} - u(0) \exp\left\{ - \int_0^{t_1} v(\tau)\, d\tau \right\}$$

This, together with (3.2.10), yields

$$u(t_1) > u(0) \exp\left\{ - \int_0^{t_1} v(\tau)\, d\tau \right\}$$

which is in contradiction with (3.2.11). The lemma has been proved.

Remark 3.2.5. The inequality

$$u(t) \le u(0) - \int_0^{t} v(\tau)\, u(\tau)\, d\tau, \qquad \text{for all } 0 \le t \le T$$

cannot imply inequality (3.2.9), even though we have the well-known Gronwa!
Bellman inequality(cf. Bellman and Cooke [1]), namely, we have

$$u(t) \le u(0) \exp\left\{ \int_0^{t} v(\tau)\, d\tau \right\}, \qquad 0 \le t \le T$$

if

$$u(t) \leq u(0) + \int_0^t v(\tau)\, u(\tau)\, d\tau, \qquad \text{for all } 0 \leq t \leq T$$

The following lemma is the useful stochastic Gronwall-Bellman inequality.

Lemma 3.2.6. Let ρ be a finite stopping time. Suppose $(A(t))_{0\leq t\leq \rho}$ is a non-decreasing continuous adapted process such that $A(0) = 0$ and $A(\rho) \leq N$ = constant a.s. and $(X(t))_{0\leq t\leq \rho}$ is a non-decreasing progressive process. If

$$E\,X(\tau) \leq x_o + E\int_0^\tau X(s)\, dA(s) \tag{3.2.12}$$

for any stopping time τ with $0 \leq \tau \leq T$, where x_o is a constant, then we have

$$E\,X(\rho) \leq x_o e^N \tag{3.2.13}$$

Proof. For arbitrarily given integer n such that $N/n < 1$, define stopping times

$$\sigma_o := 0$$

$$\sigma_i := \inf\{t > \sigma_i : A(t) - A(\sigma_i) > N/n\} \wedge T, \qquad 1 \leq i \leq n$$

Set

$$x_i = E\,X(\sigma_i), \qquad i = 1, 2, \dots, n$$

By induction, we can prove

$$x_i \leq x_o/(1-\varepsilon)^i, \qquad i = 1, 2, \dots, n \tag{3.2.14}$$

where $\varepsilon = N/n$. Indeed, we have

$$x_1 \leq x_o + E\int_0^{\sigma_1} X(s)\, dA(s) \leq x_o + E\int_0^{\sigma_1} X(\sigma_1)\, dA(s)$$

$$\leq x_o + \varepsilon\, E\,X(\sigma_1) = x_o + \varepsilon\, x_1$$

which means

$$x_1 \leq x_o/(1-\varepsilon)$$

We now suppose (3.2.14) holds for $i = 1, 2, \dots, m$. Then

$$x_{m+1} = x_0 + \sum_{i=1}^{m+1} E \int_{\sigma_{i-1}}^{\sigma_i} X(s)\, dA(s)$$

$$\leq x_0 + \sum_{i=1}^{m+1} E \int_{\sigma_{i-1}}^{\sigma_i} X(\sigma_i)\, dA(s) \leq x_0 + \sum_{i=1}^{m+1} x_i$$

It follows immediately that

$$x_{m+1} \leq \frac{x_0}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon} \sum_{i=1}^{m} x_i \leq \frac{x_0}{(1-\varepsilon)^{m+1}}$$

So we have proved (3.2.14). Noticing $\sigma_n = \rho$ a.s., we get

$$E\, X(\rho) \leq x_0 (1 - N/n)^{-n}$$

Letting $n \to \infty$, we deduce

$$EX(\rho) \leq x_0 e^N$$

which is the desired assertion.

3.3. STOCHASTIC STABILITY

Consider the following SIES

$$X = x_0 + \sum_{i=1}^{m} F_i(X).M_i \tag{3.3.1}$$

where $x_0 \in R^n$ and M_i ($1 \leq i \leq m$) are semimartingales with $M_i(0) = 0$. We always assume that F_i ($1 \leq i \leq m$) satisfy the conditions of the existence and uniqueness of the solutions discussed in Chapter I. Denote this solution by $X(x_0) = (X(x_0, t))_{t \geq 0}$. It is clear that $X(x_0)$ is a semimartingale satisfying the initial condition $X(x_0, 0) = x_0$ a.s. Furthermore, we also suppose $F_i(0) = 0$ for all $1 \leq i \leq m$. We then have $X(0) = 0$ which will be called the zero solution or the trivial solution.

Set $S_h = \{x \in R^n : |x| < h\}$ and $\underline{S}_h = \{x \in R^n : |x| \leq h\}$ for $0 < h \leq \infty$.

We now give the definitions of stochastic stability as follows:

Definition 3.3.1. The zero solution is said to be

(1) stochastically stable if for any $r > 0$ and $\varepsilon > 0$ there exists a $\delta = \delta(r, \varepsilon)$ such that for any $x_0 \in S_\delta$,

$$P\{\omega : |X(x_0, t)| < r \text{ for all } t \geq 0\} > 1 - \varepsilon \tag{3.3.2}$$

(2) stochastically asymptotically stable if the zero solution is stochastically stable as well as for any $\varepsilon > 0$ there exists a positive $\delta = \delta(\varepsilon)$ such that

$$P\{\omega : \lim_{t\to\infty} |X(x_0, t)| = 0\} > 1-\varepsilon \tag{3.3.3}$$

for any $x_0 \in S_\delta$;

(3) stochastically globally asymptotically stable if the zero solution is stochastically stable and

$$P\{\omega : \lim_{t\to\infty} |X(x_0, t)| = 0\} = 1 \tag{3.3.4}$$

for all $x_0 \in R^n$.

We now begin to prove our first theorem about the stochastic stability.

Theorem 3.3.2. Let M_i $(1\leq i\leq m) \in \mathcal{H}^\infty$ with $M_i(0) = 0$ and $F_i \in \mathcal{L}^p(\{a_k\}, K)$, where $1 \leq p < \infty$, $K > 0$ and $\{a_k\}$ is a sequence of positive real numbers. Then the zero solution is stochastically stable.

Proof. Set $0 < b < 1/(m\, C_p\, K)$. By Lemma 1.3.8, there exist a number of stopping times $\{T_k\}_{0\leq k\leq l}$ with $0 = T_0 \leq T_1 \leq \ldots \leq T_l = \infty$ a.s. such that $T_k > T_{k-1}$ on $\{T_{k-1} < \infty\}$ and

$$\| M_i^{T_k-} - M_i^{T_{k-1}} \|_{\mathcal{H}^\infty} \leq b \quad \text{for all } 1 \leq i \leq m \text{ and } 1 \leq k \leq l$$

In the same way as the proof of Corollary 1.3.9 we can prove

$$\| X(x_0) \|_{\mathcal{S}^p} = \| X(x_0)^{T_l-} \|_{\mathcal{S}^p} \leq \frac{|x_0|}{1 - m\, C_p K} \cdot \frac{\alpha^l - 1}{\alpha - 1} \tag{3.3.5}$$

for all $x_0 \in R^n$, where

$$\alpha = \frac{C_p K \sum_{i=1}^{m} \| M_i \|_{\mathcal{H}^\infty}}{1 - m\, b\, C_p K}$$

and $(\alpha^l - 1)/(\alpha - 1) = l$ if $\alpha = 1$. Therefore, using Chebyshev's inequality and Hölder's inequality we deduce, for any $r > 0$,

$$P\{\omega : \sup_{t\geq 0} |X(x_o, t)| \geq r\} \leq \frac{1}{r} \| X(x_o) \|_{\mathcal{G}^1}$$

$$\leq \frac{1}{r} \| X(x_o) \|_{\mathcal{G}^p} \leq \frac{1}{r} \cdot \frac{|x_o|}{1 - m\, C_p K} \cdot \frac{\alpha^l - 1}{\alpha - 1} \tag{3.3.6}$$

which implies that the zero solution is stochastically stable immediately. The proof is complete.

In order to give other theorems about the stochastic stability we will need some notations. Denote by $C^{2,1}(Q \times R_+)$ the family of all functions $V(x, t) : Q \times R_+ \to R_+$ which have continuous second partial derivatives in x and first partial derivative in t, where Q is an open subset of R^n. If $V(x, t) \in C^{2,1}(Q \times R_+)$, we set

$$V_t(x, t) = \frac{\partial}{\partial t} V(x, t), \qquad V_x(x, t) = \left(\frac{\partial}{\partial x_1} V(x, t), \ldots, \frac{\partial}{\partial x_n} V(x, t) \right)$$

$$\text{and} \quad V_{xx}(x, t) = \left(\frac{\partial^2}{\partial x_i \partial x_j} V(x, t) \right)_{n \times n}$$

Let $\mathcal{K}$ denote the set of all nondecreasing functions $\mu : R_+ \to R_+$ such that $\mu(r) > 0$ for all $r > 0$. Define stopping times

$$T_r = T_r(x_o) = \inf\{ t \geq 0 : |X(x_o, t)| \geq r \}$$

and

$$\tau_r = \tau_r(x_o) = \inf\{ t \geq 0 : |X(x_o, t)| \leq r \}$$

for $r > 0$ and $x_o \in R^n$. We will also need the following hypotheses.

(H1) M_i $(1 \leq i \leq m)$ are continuous semimartingales with $M_i(0) = 0$ and

$$\langle M_i^c, M_j^c\rangle(t) = \int_0^t m_{ij}(s)\, dA(s), \quad t \ge 0, \quad i, j = 1, 2, \ldots, m$$

where A is a continuous nondecreasing adapted process and m_{ij} $(1 \le i, j \le m)$ are a integrable adapted processes with respect to A.

(H2) There exist functions $V(x, t) \in C^{2,1}(S_h \times R_+)$ $(h > 0)$, $G : \mathcal{X} \to \mathcal{P}$ an $\mu \in \mathcal{K}$ satisfying the following conditions:

(1) $\mu(|x|) \le V(x, t)$ for all $(x, t) \in R^n \times R_+$;

(2) $V(0, 0) = 0$;

(3) for any $x_0 \in S_h$ and any bounded stopping time S,

$$E\left\{\sum_{i=1}^{m} \int_0^{S \wedge T_h} V_x\big(X(x_0, s), A(s)\big)\, F_i(X(x_0))(s)\, dM_i(s)\right\}$$

$$\le E \int_0^{S \wedge T_h} G(X(x_0))(s)\, dA(s) < \infty$$

provided $\int_0^{S \wedge T_h} V_x\big(X(x_0, s), A(s)\big)\, F_i(X(x_0))(s)\, dM_i(s)$ $(1 \le i \le m)$ are integrable;

(4) for any $x_0 \in S_h$ and $t \ge 0$,

$$\mathbb{L}\, V(X(x_0, t), t)\, \mathbb{1}_{]\!]0, T_h]\!]}(t) \le 0 \qquad \text{a.s.}$$

where

$$\mathbb{L}\, V(X(x_0, t), t) := V_t(X(x_0, t), t) + G(X(x_0))(t)$$

$$+ \frac{1}{2}\sum_{i,j=1}^{m} m_{ij}(t)\, (F_i(X(x_0))(t))'\, V_{xx}(X(x_0, t), t)\, F_j(X(x_0))(t)$$

(H3) There exist functions $V(x, t) \in C^{2,1}(S_h \times R_+)$ $(h > 0)$, $G : \mathcal{X} \to \mathcal{P}$ and $\mu, \gamma \in \mathcal{K}$ satisfying the following conditions:

(1) $\mu(|x|) \le V(x, t)$ for all $(x, t) \in R^n \times R_+$;

(2) $\lim_{x \to 0} V(x, t) = 0$ uniformly in $t \in R_+$;

(3) for any $x_o \in S_h$ and any bounded stopping times S and T with $S \leq T$ a.s.,

$$E\left\{ \sum_{i=1}^{m} \int_{S \wedge T_h}^{T \wedge T_h} V_x\big(X(x_o, s), A(s)\big) F_i(X(x_o))(s)\, dM_i(s) \right\}$$

$$\leq E \int_{S \wedge T_h}^{T \wedge T_h} G(X(x_o))(s)\, dA(s) < \infty$$

provided $\int_{S \wedge T_h}^{T \wedge T_h} V_x\big(X(x_o, s), A(s)\big) F_i(X(x_o))(s)\, dM_i(s)$ $(1 \leq i \leq m)$ are integrable;

(4) for any $\alpha \in (0, h)$, $x_o \in S_h - \underline{S}_\alpha$ and $t \geq 0$,

$$\mathbb{L}\, V(X(x_o, t), t)\, \mathbb{1}_{]\!]0,\, \tau_\alpha \wedge T_h]\!]}(t) \leq -\gamma(\alpha)\, \mathbb{1}_{]\!]0,\, \tau_\alpha \wedge T_h]\!]}(t) \qquad \text{a.s.}$$

We now have the following several theorems about the stochastic stability.

Theorem 3.3.3. Assume hypotheses (H1) and (H2) hold. Then the zero solution is stochastically stable.

Proof. It is clear that for any $\varepsilon > 0$ and $r > 0$ $(r < h)$ there exists a $\delta = \delta(\varepsilon, r)$ with $0 < \delta < r$ such that

$$\frac{1}{\varepsilon} \sup_{x_o \in S_\delta} V(x_o, 0) \leq \mu(r) \tag{3.3.7}$$

We now let $x_o \in S_h$ be arbitrary and define

$$U_k = k \wedge \inf\left\{ t \geq 0 : \left| \sum_{i=1}^{m} \int_0^{t \wedge T_h} V_x\big(X(x_o, s), A(s)\big) F_i(X(x_o))(s)\, dM_i(s) \right| \geq k \right\}$$

for $k = 1, 2, \ldots$. It is obvious that U_k is a bounded stopping time and $U_k \uparrow \infty$ a.s. Using Itô's formula and hypotheses (H2) we get

$$E\,V(X(x_0, T_r \wedge U_k),\ A(T_r \wedge U_k))$$

$$\leq\ V(x_0, 0) + E \int_0^{T_h \wedge U_k} \mathbb{L}\, V(X(x_0, s), s)\, dA(s) \ \leq\ V(x_0, 0)$$

By Fatou's convergence lemma and letting $k \to \infty$, we get

$$E\left\{ V(X(x_0, T_r),\ A(T_r))\, \mathbb{1}_{\{T_r < \infty\}} \right\}\ \leq\ V(x_0, 0)$$

However, we also have

$$E\left\{ V(X(x_0, T_r),\ A(T_r))\, \mathbb{1}_{\{T_r < \infty\}} \right\}\ \geq\ \mu(r)\, P\{ T_r < \infty \}$$

These, together with (3.3.7), imply

$$P\{ T_r < \infty \}\ \leq\ \varepsilon$$

i.e.

$$P\{ | X(x_0, t) | < r \text{ for all } t \geq 0 \}\ \geq\ 1 - \varepsilon$$

which completes the proof.

Theorem 3.3.4. Let hypotheses (H1) and (H3) hold. Assume

(H4) $$\lim_{t \to \infty} A(t) \ =\ \infty \quad \text{a.s.}$$

Then the zero solution is stochastically asymptotically stable.

Proof. By Theorem 3.3.3, for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$P\{ | X(x_0, t) | < h \text{ for all } t \geq 0 \}\ \geq\ 1 - \varepsilon/5, \qquad x_0 \in S_\delta \tag{3.3.8}$$

For any fixed $x_0 \in S_\delta$, let $0 < \alpha < \beta < | x_0 |$ arbitrarily. Define

$$U_k = k \wedge \inf\left\{ t \geq 0 : \left| \sum_{i=1}^{m} \int_0^{t \wedge \tau_\alpha \wedge T_h} V_x\big(X(x_0, s), A(s)\big) \times \right.\right.$$

$$\left.\left. \times F_i(X(x_0))(s)\, dM_i(s) \right| \geq k \right\}$$

for $k = 1, 2, \ldots$. Clearly, U_k is a bounded stopping time and $U_k \uparrow \infty$ a.s. Using Itô's formula and hypothesis (H3) we deduce

$$0\ \leq\ E\,V(X(x_0, T_r \wedge S_k),\ A(T_r \wedge S_k))\ \leq$$

$$\leq V(x_o, 0) + E \int_0^{\tau_\alpha \wedge T_h \wedge U_k} \mathbb{L} V(X(x_o, s), s)\, dA(s) \;\leq V(x \quad 0)$$

$$\leq V(x_o, 0) \; - \; \gamma(\alpha)\, E\, A(\tau_\alpha \wedge T_h \wedge U_k)$$

Consequently

$$E\, A(\tau_\alpha \wedge T_h \wedge U_k) \;\leq\; V(x_o, 0)/\gamma(\alpha) \tag{3.3.9}$$

which, together with (H4), implies

$$P\{\tau_\alpha \wedge T_h < \infty\} \;=\; 1$$

Notice, by (3.3.8),

$$P\{\tau_\alpha \wedge T_h < \infty\} \;\leq\; P\{\tau_\alpha < \infty\} + P\{T_h < \infty\}$$
$$\leq\; P\{\tau_\alpha < \infty\} + \varepsilon/5$$

Thus

$$P\{\tau_\alpha < \infty\} \;\geq\; 1 - \varepsilon/5$$

Hence, there exists a positive constant $\theta = \theta(\alpha)$ such that

$$P\{\tau_\alpha < \theta\} \;\geq\; 1 - 2\varepsilon/5$$

Therefore

$$P\{\tau_\alpha < T_h \wedge \theta\} \geq P(\{\tau_\alpha < \theta\} \cap \{T_h = \infty\})$$

$$\geq P\{\tau_\alpha < \theta\} - P\{T_h < \infty\} \;\geq 1 - 3\varepsilon/5 \tag{3.3.10}$$

Define the stopping times

$$\xi = \begin{cases} \tau_\alpha, & \text{if } \omega \in \{\tau_\alpha < \theta \wedge T_h\} \\ \infty, & \text{otherwise} \end{cases}$$

$$\zeta = \inf\{t > \xi : |X(x_o, t)| \geq \beta\}$$

and

$$\Gamma_i = \inf\Big\{ t \geq \xi : \Big| \sum_{i=1}^{m} \int_{\xi}^{t \wedge \zeta} V_x\big(X(x_o, s), A(s)\big)\, F_i(X(x_o))(s)\, dM_i \quad \Big| \geq i \Big\}$$

for $i = 1, 2, \ldots$. Clearly, $\Gamma_i \uparrow \infty$ a.s. By Itô's formula and hypo sis (H3) we can similarly deduce

$$E\,V\big(X(x_0,\,\xi\wedge T_h\wedge t),\,A(\xi\wedge T_h\wedge t)\big)$$
$$\geq\; E\,V\big(X(x_0,\,\zeta\wedge\Gamma_i\wedge T_h\wedge t),\,A(\zeta\wedge\Gamma_i\wedge T_h\wedge t)\big)$$

for all $i = 1, 2, \ldots$ and $t > 0$. Letting $i \to \infty$ and $t \to \infty$, we obtain

$$E\left\{V\big(X(x_0,\,\xi\wedge T_h),\,A(\xi\wedge T_h)\big)\,\mathbb{1}_{\{\xi<\infty\}}\right\}$$
$$\geq\; E\left\{V\big(X(x_0,\,\zeta\wedge T_h),\,A(\zeta\wedge T_h)\big)\,\mathbb{1}_{\{\zeta<\infty\}}\right\}$$

Consequently

$$E\left\{V\big(X(x_0,\,\tau_\alpha),\,A(\tau_\alpha)\big)\,\mathbb{1}_{\{\tau_\alpha<\theta\wedge T_h\}}\right\}$$
$$\geq\; E\left\{V(X(x_0,\,\zeta),\,A(\zeta))\,\mathbb{1}_{\{\zeta<\infty\}\cap\{T_h=\infty\}}\right\} \qquad (3.3.11)$$

We now let

$$B_\alpha \;=\; \sup\{\,V(x,t) \,:\, (x,t)\in \underline{S}_\alpha\times R_+\,\}$$

By hypothesis (2) of (H3) we have $\lim_{\alpha\to 0} B_\alpha = 0$. Hence, we can let α so small that

$$B_\alpha/\gamma(\beta) \;<\; \varepsilon/5 \qquad (3.3.12)$$

Then, it follows from (3.3.11) that

$$P(\{\zeta<\infty\}\cap\{T_h=\infty\}) \;\leq\; B_\alpha/\gamma(\beta) \;<\; \varepsilon/5$$

On the other hand,

$$P(\{\zeta<\infty\}\cap\{T_h=\infty\}) \;\geq\; P\{\zeta<\infty\} - P\{T_h<\infty\}$$
$$\geq\; P\{\zeta<\infty\} - \varepsilon/5$$

Therefore

$$P\{\zeta<\infty\} \;<\; 2\varepsilon/5$$

Consequently, by (3.3.10),

$$P\{\xi<\infty \text{ and } \zeta=\infty\} \;\geq\; P\{\xi<\infty\} - P\{\zeta<\infty\}$$
$$\geq\; P\{\tau_\alpha < T_h\wedge\theta\} - 2\varepsilon/5 \;>\; 1-\varepsilon$$

which means

$$P\{\,\omega : \overline{\lim_{t\to\infty}}\,|X(x_0,t)| \;\leq\; \beta\,\} \;>\; 1-\varepsilon$$

Since β is arbitrary we get

$$P\{\omega : \lim_{t\to\infty} |X(x_0, t)| = 0\} \geq 1 - \varepsilon$$

The proof is complete.

The following theorem can be proved in the same way as Theorem 3.3.4.

Theorem 3.3.5. Let hypotheses (H1) and (H3) with $h = \infty$ hold. Assume

$$\text{(H4)} \qquad \lim_{t\to\infty} A(t) = \infty \quad \text{a.s.}$$

Then the zero solution is stochastically globally asymptotically stable.

We now give some examples to illustrate our results.

Example 3.3.6. Consider a one-dimensional linear SIES

$$X(t) = x_0 + \int_0^t a(s)\, X(s)\, dA(s) + \int_0^t b(s)\, X(s)\, dM(s), \qquad t \geq 0 \quad (3.3.13)$$

where $x_0 \in R$, A is a one-dimensional continuous nondecreasing adapted process such that $A(0) = 0$ and

$$\lim_{t\to\infty} A(t) = \infty \quad \text{a.s.} \qquad (3.3.14)$$

M is a one-dimensional continuous martingale such that $M(0) = 0$ and

$$\langle M, M\rangle(t) = \int_0^t m(s)\, dA(s) \quad (t \geq 0) \qquad (3.3.15)$$

Assume a, b and m are all bounded predictable processes. Define

$$V(x, t) = x^2 \quad \text{and} \quad G(X(x_0))(t) = a(t)\, V_x(X(x_0, t), A(t))\, X(x_0, t)$$

It is obvious that for any bounded stopping times S and T with $S \leq T$ a.s. we have

$$E\int_S^T V_x(X(x_0, s), A(s)) \left\{ a(s)\, X(x_0, s)\, dA(s) + b(s)\, X(x_0, s)\, dM(s) \right\} =$$

$$= E \int_S^T G(X(x_o))(s)\, dA(s)$$

provided $\int_S^T V_x(X(x_o, s), A(s)) \left\{ a(s)\, X(x_o, s)\, dA(s) + b(s)\, X(x_o, s)\, dM(s) \right\}$ is integrable. We also have

$$\mathbb{L}\, V(X(x_o, t), t) = \left\{ 2\, a(t) + b^2(t)\, m(t) \right\} X^2(x_o, t)$$

Hence, if $2\, a(t) + b^2(t)\, m(t) \le 0$ a.s. for all $t \ge 0$, Theorem 3.3.3 yields the zero solution of Eq.(3.3.13) is stochastically stable. if $2\, a(t) + b^2(t)\, m(t) \le -c$ a.s. for all $t \ge 0$, where c is a positive constant, Theorem 3.3.5 yields the zero solution of Eq.(3.3.13) is stochastically globally asymptotically stable.

Example 3.3.6. Consider the following two-dimensional SIES

$$X = x_o + F_1(X).A + F_2(X).M \tag{3.3.16}$$

where $x_o \in R^2$, A is a one-dimensional continuous nondecreasing adapted process with $A(0) = 0$ and M is a one-dimensional continuous martingale such that $M(0) = 0$ and

$$\langle M, M\rangle(t) = \int_0^t s\, dA(s) \quad (t \ge 0) \tag{3.3.17}$$

and

$$F_1(X)(t) = \begin{bmatrix} -\dfrac{X_1(t)}{1+t} + t^2 X_2(t) \\[2ex] -\dfrac{X_2(t)}{1+t} - t^2 X_1(t) \end{bmatrix}, \qquad F_2(X)(t) = \begin{bmatrix} -\dfrac{X_2(t)}{1+t} \\[2ex] \dfrac{X_1(t)}{1+t} \end{bmatrix}$$

Let

$$V(x, t) = (1+t)\,|x|^2$$

and

$$G(X(x_o))(t) = V_x(X(x_o, t), A(t))\, F_1(X(x_o))(t)$$

We then have, for any bounded stopping times S and T with $S \le T$ a.s., that

$$E\int_S^T V_x(X(x_o, s), A(s))\left\{F_1(X(x_o))(s).dA(s) + F_2(X(x_o))(s).dM(s)\right\}$$

$$= E\int_S^T G(X(x_o)(s)\, dA(s)$$

provided $\int_S^T V_x(X(x_o, s), A(s))\left\{F_1(X(x_o))(s)\, dA(s) + F_2(X(x_o))(s)\, dM(s)\right\}$ is integrable. We also have

$$\mathbb{L}\, V(X(x_o, t), t) = |X(x_o, t)|^2 - 2\,|X(x_o, t)|^2 + \frac{t\,(1+t)}{(1+t)^2}|X(x_o, t)|^2 \le 0$$

Therefore, by Theorem 3.3.3, the zero solution of Eq.(3.3.16) is stochastically stable.

3.4. STOCHASTIC BOUNDEDNESS

In this section we will consider the same Eq.(3.3.1). However we do not require $F_i(0) = 0$ $(1\le i\le m)$. We first give the definitions of stochastic boundedness.

Definition 3.4.1. The solutions of Eq.(3.3.1) will be said to be

(1) stochastically bounded if for any $\delta > 0$ and $\varepsilon > 0$ there exists a positive number $b = b(\delta, \varepsilon)$ such that for any $x_o \in S_\delta$,

$$P\{\omega : |X(x_o, t)| < b \text{ for all } t \ge 0\} > 1-\varepsilon \quad (3.4.1)$$

(2) stochastically ultimately bounded if the solutions are stochastically bounded as well as for any $\varepsilon > 0$ there exists a positive $b = b(\varepsilon)$ such that

$$P\{\omega : \overline{\lim_{t\to\infty}}\, |X(x_o, t)| \le b\} > 1-\varepsilon \quad (3.4.2)$$

for any $x_o \in R^n$.

We denote by $\mathcal{V}$ the set of all nonincreasing functions $v : (0, \infty) \to (0, \infty)$. We also need following hypotheses.

(H5) There exist functions $V(x, t) \in C^{2,1}(R^n \times R_+)$, $G : \mathcal{X} \to \mathcal{P}$ satisfying the following conditions:

(1) $0 \le V(x, t)$ for all $(x, t) \in R^n \times R_+$;

(2) $\lim_{|x|\to\infty} V(x, t) = \infty$ uniformly in $t \in R_+$;

(3) for any $x_0 \in R^n$ and bounded stopping time S,

$$E\left\{\sum_{i=1}^{m}\int_0^S V_x\big(X(x_0, s), A(s)\big)\, F_i(X(x_0))(s)\, dM_i(s)\right\}$$

$$\le E\int_0^S G(X(x_0))(s)\, dA(s) < \infty$$

provided $\int_0^S V_x\big(X(x_0, s), A(s)\big)\, F_i(X(x_0))(s)\, dM_i(s)$ $(1 \le i \le m)$ are integrable;

(4) for any $x_0 \in R^n$ and $t \ge 0$,

$$\mathbb{L}\, V(X(x_0, t), t) \le 0 \qquad \text{a.s.}$$

(H6) There exist functions $U(x, t) \in C^{2,1}((R^n - S_\alpha) \times R_+)$ $(0<\alpha<\infty)$, $F : \mathcal{X} \to \mathcal{P}$ and $v \in \mathcal{V}$ which satisfy the following conditions:

(1) $0 \le U(x, t)$ for all $(x, t) \in (R^n - S_\alpha) \times R_+$;

(2) $\lim_{|x|\to\infty} U(x, t) = \infty$ uniformly in $t \in R_+$;

(3) for any $x_0 \in (R^n - S_\alpha) \times R_+$ and bounded stopping time S,

$$E\left\{\sum_{i=1}^{m}\int_0^S U_x\big(X(x_0, s), A(s)\big)\, F_i(X(x_0))(s)\, dM_i(s)\right\}$$

$$\leq E \int_0^S F(X(x_o))(s)\, dA(s) \;<\; \infty$$

provided $\int_S^T U_x\big(X(x_o, s), A(s)\big)\, F_i(X(x_o))(s)\, dM_i(s)$ $(1 \leq i \leq m)$ are integrable;

(4) for any $h \in (\alpha, \infty)$, $x_o \in S_h - \underline{S}_\alpha$ and $t \geq 0$,

$$\mathcal{L}U(X(x_o, t), t)\, \mathbb{1}_{]\!]0,\, \tau_\alpha \wedge T_h]\!]}(t) \;\leq\; -\, v(h)\, \mathbb{1}_{]\!]0,\, \tau_\alpha \wedge T_h]\!]}(t) \qquad \text{a.s.}$$

where

$$\mathcal{L}U(X(x_o, t), t) \;=\; U_t(X(x_o, t), t) \;+\; F(X(x_o))(t)$$

$$+\; \frac{1}{2} \sum_{i,j=1}^{m} m_{ij}(t)\, \big(F_i(X(x_o))(t)\big)'\, U_{xx}(X(x_o, t), t)\, F_j(X(x_o))(t)$$

(H7) There exist functions $V(x, t) \in C^{2,1}(R^n \times R_+)$ $(h > 0)$, $G : \mathcal{X} \rightarrow \mathcal{P}$ and $\eta \in \mathcal{K}$ satisfying the following conditions:

(1) $0 \leq V(x, t) \leq \eta(|x|)$ for all $(x, t) \in R^n \times R_+$;

(2) $\lim\limits_{|x| \mapsto \infty} V(x, t) = \infty$ uniformly in $t \in R_+$;

(3) for any $x_o \in R^n$ and bounded stopping times S and T with $S \leq T$ a.s.,

$$E\Big\{ \sum_{i=1}^{m} \int_S^T V_x\big(X(x_o, s), A(s)\big)\, F_i(X(x_o))(s)\, dM_i(s) \Big\}$$

$$\leq E \int_S^T G(X(x_o))(s)\, dA(s) \;<\; \infty$$

provided $\int_S^T V_x\big(X(x_o, s), A(s)\big)\, F_i(X(x_o))(s)\, dM_i(s)$ $(1 \leq i \leq m)$ are integrable;

(4) for any $x_o \in R^n$ and $t \geq 0$,

$$\mathbb{L} V(X(x_o, t), t) \;\leq\; 0 \qquad \text{a.s.}$$

We now begin to prove our theorems about the stochastic boundedness.

Theorem 3.4.2. Assume (H1) and (H5) hold. Then the solutions of Eq.(3.3.1) are stochastic bounded.

Proof. For any $\delta > 0$, set

$$K(\delta) \;=\; \sup_{x \in S_\delta} V(x, 0)$$

Clearly, $K(\delta) < \infty$. By conditon (2) of (H4), for any $\varepsilon > 0$ there exists a positive number $b = b(\varepsilon, \delta) > \delta$ such that

$$V(x, t) \;>\; K(\delta)/\varepsilon \qquad \text{for all } (x, t) \in (R^n - S_b) \times R_+ \tag{3.4.3}$$

In the same way as the proof of Theorem 3.3.3 we can prove

$$\begin{aligned} V(x_o, 0) \;&\geq\; E\left\{ V(X(x_o, T_b),\; A(T_b))\, \mathbb{1}_{\{T_b < \infty\}} \right\} \\ &\geq\; \mu(r)\, P\{ T_b < \infty \} \end{aligned}$$

for all $x_o \in S_\delta$, which, together with (3.4.3), implies

$$P\{ T_b < \infty \} \;\leq\; \varepsilon$$

i.e.

$$P\{ |X(x_o, t)| < b \text{ for all } t \geq 0 \} \;\geq\; 1 - \varepsilon$$

which completes the proof.

Theorem 3.4.3. Assume (H1), (H4), (H6) and (H7) hold. Then the solutions of Eq.(3.3.1) are stochastically ultimately bounded.

Proof. We first prove that for any $\in R^n$,

$$P\{ \tau_\alpha < \infty \} \;=\; 1 \tag{3.4.4}$$

It is enough to prove it for $x_o \in R^n - S_\alpha$. For any $x_o \in R^n - S_\alpha$, let $h > |x_o|$. By condition (H6) we can similarly prove

$$0 \;\leq\; E\, U(X(x_o, \tau_\alpha \wedge T_h \wedge t),\, A(\tau_\alpha \wedge T_h \wedge t))$$

$$\leq U(x_o, 0) - v(h)\, E\, A(\tau_\alpha \wedge T_h \wedge t) \tag{3.4.5}$$

for all $t \geq 0$. It follows immediately from (3.4.5) and condition (H4) that

$$P\{\tau_\alpha \wedge T_h < \infty\} = 1 \tag{3.4.6}$$

On the other hand, it also follows from (3.4.5) and (3.4.6) that

$$\begin{aligned} U(x_o, 0) &\geq E\, U(X(x_o, \tau_\alpha \wedge T_h), A(\tau_\alpha \wedge T_h)) \\ &\geq E\left\{ U(X(x_o, T_h), A(T_h))\, \mathbb{1}_{\{T_h < \tau_\alpha\}} \right\} \end{aligned}$$

By condition (2) of (H6) we deduce

$$P\{T_h < \tau_\alpha\} = 0$$

which, together with (3.4.6), implies desired (3.4.4).

We now prove condition (2) of Definition 3.4.1. Let $x_o \in R^n$ be arbitrary. For any $\varepsilon > 0$, by condition (2) of (H7) there exists a number $b(\varepsilon) > \alpha$ such that

$$V(x, t) > \eta(\alpha)/\varepsilon \quad \text{for all } (x, t) \in (R^n - S_b) \times R_+ \tag{3.4.7}$$

Define stopping time

$$\xi = \inf\{t \geq \tau_\alpha : |X(x_o, t)| \geq b\}$$

Using condition (H7) we can prove

$$E\, V(X(x_o, \xi \wedge t), A(\xi \wedge t)) \leq E\, V(X(x_o, \tau_\alpha \wedge t), A(\tau_\alpha \wedge t))$$

for any $t > 0$. Thus, it follows by letting $t \to \infty$ and using Fatou's convergence lemma that

$$E\, V(X(x_o, \xi), A(\xi))\, \mathbb{1}_{\{\xi < \infty\}} \leq E\, V(X(x_o, \tau_\alpha), A(\tau_\alpha))$$

which and (3.4.7) yield

$$P\{\xi < \infty\} < \varepsilon$$

This means

$$P\{\omega : \overline{\lim_{t \to \infty}} |X(x_o, t)| \leq b\} > 1 - \varepsilon$$

We have completed the proof since the solutions are stochastically bounded by Theorem 3.4.2.

3.5. EVENTUAL ASYMPTOTIC STABILITY

In this section we will apply the Lebesgue-Stieltjes integral inequalities obtained in section 3.2 to prove some sufficient conditions for the eventual asymptotic stabilities of solutions of weakly nonlinear SIES. Let N is a continuous local martingale with $\langle N \rangle = \langle N, N \rangle$ determinate (non-stochastic). For the convenience, we use $N_t = N(t)$ and $\langle N \rangle_t = \langle N \rangle(t)$. Consider the following SIES

$$X(t) = x_o + \int_{t_o}^{t} A\, X(s)\, d\langle N \rangle_s + \int_{t_o}^{t} f(X(s), s)\, d\langle N \rangle_s$$

$$+ \int_{t_o}^{t} G(X(s), s)\, dN_s + \int_{t_o}^{t} g(X(s), s)\, dN_s \qquad (3.5.1)$$

where $t \geq t_o \geq 0$, x_o is an n-dimensional $\mathcal{F}_{t_o}$-measurable random variable with $E\,|x_o|^2 < \infty$, $A = (a_{ij})_{n \times n}$ is a constant matrix, $f, G, g \in C(R^n \times R_+, R^n)$ are sufficiently smooth for the existence and uniqueness of the solutions discussed in Chapter I.

Let $X(t, t_o, x_o)$ denote the solutions of Eq.(3.5.1) existing in the right of t_o. We assume $E\,|X(t, t_o, x_o)|^2 < \infty$. We will also assume that $f(0, t) = G(0, t) = g(0, t) = 0$ for all $t \geq 0$. So Eq.(3.5.1) admits a trivial solution or zero solution. We first introduce the concept of the eventual asymptotic stability in mean square.

Definition 3.5.1. The trivial solution of Eq.(3.5.1) is said to be eventually uniformly asymptotically stable in mean square if

(1) for arbitrary $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $\sigma = \sigma(\varepsilon) > 0$ such that $E\,|X(t, t_o, x_o)|^2 < \varepsilon$ for all $t \geq t_o$ provided $E\,|x_o|^2 \leq \delta(\varepsilon)$ and $t_o \geq \sigma(\varepsilon)$;

(2) there exist two positive numbers δ_o and τ_o such that for arbitrary $\eta > 0$ there exists a $T = T(\eta)$ such that $E\,|X(t, t_o, x_o)|^2 < \eta$ for all $t \geq t_o + T$ provided $E\,|x_o|^2 \leq \delta_o$ and $t_o \geq \tau_o$.

We then have the following theorems.

Theorem 3.5.2. Assume that

(1) all characteristic roots of A have negative real parts;

(2) $\alpha \le \inf_{t\ge 0} (\langle N\rangle_{t+1} - \langle N\rangle_t) \le \sup_{t\ge 0} (\langle N\rangle_{t+1} - \langle N\rangle_t) \le \beta$

where α and β are strictly positive constants;

(3) for arbitrary $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that

$$|G(x,t)|^2 \le \varepsilon |x|^2 \quad \text{for all } t \ge T(\varepsilon) \text{ and } x \in R^n$$

(4) $|f(x,t)|^2 \vee |g(x,t)|^2 \le \gamma(t)$ for all $x \in R^n$ and $t \ge 0$, where $\gamma : R_+ \to R_+$ is a continuous function such that $t\,\Gamma(t) \to 0$ as $t \to \infty$, here

$$\Gamma(t) = \int_t^{t+1} \gamma(s)\, d\langle N\rangle_s$$

Then the trivial solution of Eq.(3.5.1) is eventually uniformly asymptotically stable in mean square. In particular, we have

$$\lim_{t\to\infty} E\,|X(t, t_0, x_0)|^2 = 0 \quad \text{if } E\,|x_0|^2 < \infty \text{ and } t_0 \ge 0$$

Proof. Set $X(t) = X(t, t_0, x_0)$. By Theorem 2.4.1, we have

$$\begin{aligned} X(t) = {} & \exp(A (\langle N\rangle_t - \langle N\rangle_{t_0}))\, x_0 \\ & + \int_{t_0}^{t} \exp(A (\langle N\rangle_t - \langle N\rangle_s))\, f(X(s), s)\, d\langle N\rangle_s \\ & + \int_{t_0}^{t} \exp(A (\langle N\rangle_t - \langle N\rangle_s))\, G(X(s), s)\, dN_s \\ & + \int_{t_0}^{t} \exp(A (\langle N\rangle_t - \langle N\rangle_s))\, g(X(s), s)\, dN_s \end{aligned} \tag{3.5.2}$$

By hypothesis (1), there exist two positive constants a and c such that

$$| \exp(A (\langle N\rangle_t - \langle N\rangle_{t_o}))|^2 \le \frac{c}{4} \exp(- a (\langle N\rangle_t - \langle N\rangle_{t_o})) \qquad (3.5.3)$$

for all $t \ge t_o \ge 0$. Choose $T = \max\{ T(a/2c), 1 \}$. Let $t_o \ge T$ and $E |x_o|^2 < \infty$. For $t \ge t_o$, we have

$$E | X(t) |^2 \le c E |x_o|^2 \exp(- a (\langle N\rangle_t - \langle N\rangle_{t_o}))$$

$$+ \frac{a}{2} \int_{t_o}^{t} \exp(- a (\langle N\rangle_t - \langle N\rangle_s)) E | X(s) |^2 d\langle N\rangle_s$$

$$+ c (\langle N\rangle_t - \langle N\rangle_{t_o} + 1) \int_{t_o}^{t} \exp(- a (\langle N\rangle_t - \langle N\rangle_s)) \gamma(s) d\langle N\rangle_s$$

here the equality

$$E \left| \int_{t_o}^{t} u_s dN_s \right|^2 = E \int_{t_o}^{t} u_s^2 d\langle N\rangle_s$$

(cf. Yan [1]) has been used. Hence

$$\exp(a \langle N\rangle_t) E | X(t) |^2 \le c E |x_o|^2 \exp(a \langle N\rangle_{t_o})$$

$$+ \frac{a}{2} \int_{t_o}^{t} \exp(a \langle N\rangle_s) E | X(s) |^2 d\langle N\rangle_s$$

$$+ c (\langle N\rangle_t - \langle N\rangle_{t_o} + 1) \int_{t_o}^{t} \exp(a \langle N\rangle_s) \gamma(s) d\langle N\rangle_s$$

By Lemma 3.2.2, we then have

$$E | X(t) |^2 \le c E |x_o|^2 \exp(- \frac{a}{2} (\langle N\rangle_t - \langle N\rangle_{t_o})) +$$

$$+ c\,(\langle N\rangle_t - \langle N\rangle_{t_o} + 1) \int_{t_o}^{t} \exp\!\left(-\frac{a}{2}\,(\langle N\rangle_t - \langle N\rangle_s)\right) \gamma(s)\, d\langle N\rangle_s \qquad (3.5.4)$$

However, by hypothesis (2), we have

$$(t - s - 1)\,\alpha \;\le\; \langle N\rangle_t - \langle N\rangle_s \;\le\; (t - s + 1)\,\beta$$

for all $t \ge s \ge 0$. Thus it follows form (3.5.4) that

$$E\,|X(t)|^2 \le c\,E\,|x_0|^2 \exp\!\left(-\frac{a\alpha}{2}\,(t - t_o - 1)\right)$$

$$+ (c + c\beta(t - t_o + 1)) \int_{t_o}^{t} \exp\!\left(-\frac{a\alpha}{2}\,(t - s - 1)\right) \gamma(s)\, d\langle N\rangle_s \qquad (3.5.5)$$

By changing the order of the integration, we have

$$\int_{t_o-1}^{t} \exp\!\left(\frac{a\alpha}{2}\,(s+1)\right) \Gamma(s)\, ds$$

$$= \int_{t_o-1}^{t} \exp\!\left(\frac{a\alpha}{2}\,(s+1)\right) \left\{ \int_{s}^{s+1} \gamma(v)\, d\langle N\rangle_v \right\} ds$$

$$\ge \int_{t_o}^{t} \left\{ \int_{v-1}^{v} \exp\!\left(\frac{a\alpha}{2}\,(s+1)\right) ds \right\} \gamma(v)\, d\langle N\rangle_v$$

$$\ge \int_{t_o}^{t} \frac{2}{a\alpha}\,(e^{a\alpha/2} - 1)\, e^{a\alpha v/2}\, \gamma(v)\, d\langle N\rangle_v$$

$$\ge \int_{t_o}^{t} e^{a\alpha s/2}\, \gamma(s)\, d\langle N\rangle_s$$

Hence, we deduce from (3.5.5) that

$$E\,|X(t)|^2 \le c\,E\,|x_o|^2 \exp\left(-\frac{a\alpha}{2}(t-t_o-1)\right)$$

$$+\,(c + c\beta(t-t_o+1))\,e^{a\alpha}\int_{t_o-1}^{t} \exp\left(-\frac{a\alpha}{2}(t-s)\right)\Gamma(s)\,ds \qquad (3.5.6)$$

But

$$\int_{t_o-1}^{t} \exp\left(-\frac{a\alpha}{2}(t-s)\right)\Gamma(s)\,ds$$

$$\le \int_{0}^{t/2} \exp\left(-\frac{a\alpha}{2}(t-s)\right)\Gamma(s)\,ds + \int_{t/2}^{t} \exp\left(-\frac{a\alpha}{2}(t-s)\right)\Gamma(s)\,ds$$

$$\le \Phi(1)\frac{2}{a\alpha}e^{-a\alpha t/4} + \Phi(t/2+1)\frac{2}{a\alpha}$$

where $\Phi(t) = \sup\{\Gamma(s) : t-1 \le s < \infty\}$ for $t \ge 1$. Hence, we have that

$$E\,|X(t)|^2 \le c\,E\,|x_o|^2 \exp\left(-\frac{a\alpha}{2}(t-t_o-1)\right) +$$

$$+\,\frac{2}{a\alpha}c\,(1 + \beta(t-t_o+1))\,e^{a\alpha}\left\{\Phi(1)\,e^{-a\alpha t/4} + \Phi(t/2+1)\right\}$$

$$\to 0 \qquad \text{as } t \to \infty$$

here condition (4) has been used. This proves the trivial solution of Eq.(3.5.1) is eventually uniformly asymptotically stable in mean square and that

$$\lim_{t\to\infty} E\,|X(t, t_o, x_o)|^2 = 0 \quad \text{if } E\,|x_o|^2 < \infty \text{ and } t_o \ge T$$

Hence, for any $t_o \ge 0$ and $E\,|x_o|^2 < \infty$, we have

$$\lim_{t\to\infty} E\,|X(t, t_o, x_o)|^2 = \lim_{t\to\infty} E\,|X(t, T, X(T, t_o, x_o))|^2 = 0$$

since we have $E\,|X(t, t_o, x_o)|^2 < \infty$. This completes the proof.

Theorem 3.5.3. Assume that all the conditions in Theorem 3.5.2 are satisfied except condition (3) which is replaced by

(3') $|G(x,t)|^2 \le \lambda(t)\, w(|x|^2)$ for all $x \in R^n$ and $t \ge 0$, where w is a nonnegative nondecreasing concave function such that $w(v) > 0$ for $v > 0$ and

$$\int_{0+} \frac{dt}{w(t)}$$

is divergent, and $\lambda(t)$ is a nonnegative integrable function with respect to $\langle N\rangle$ such that

$$\int_0^\infty \lambda(s)\, d\langle N\rangle_s < \infty$$

Then the trivial solution of Eq.(3.5.1) is eventually uniformly asymptotically stable in mean square.

Proof. Using (3.5.2), (3.5.3) and the assumptions of the theorem, we have

$$E|X(t)|^2 \le c\,E|x_0|^2 + c\int_{t_0}^{t} \lambda(s)\, w(E|X(s)|^2)\, d\langle N\rangle_s$$

$$+ (c + c\beta(t - t_0 + 1)) \int_{t_0}^{t} \exp(-a\alpha(t - s - 1))\, \gamma(s)\, d\langle N\rangle_s \tag{3.5.7}$$

for all $t \ge t_0 \ge 0$ and $E|x_0|^2 < \infty$. If $t \ge t_0 \ge 1$, it is similar to the proof of Theorem 3.1 to prove

$$(c + c\beta(t - t_0 + 1)) \int_{t_0}^{t} \exp(-a\alpha(t - s - 1))\, \gamma(s)\, d\langle N\rangle_s$$

$$\le \frac{c}{a\alpha}(1 + \beta(t - t_0 + 1))\, e^{2a\alpha}\left\{\Phi(1)\, e^{-a\alpha t/2} + \Phi(t/2 + 1)\right\}$$

$$\le \Phi^*(t_0) \tag{3.5.8}$$

where

$$\Phi^*(t_0) = \sup_{t \ge t_0} \Big(\frac{c}{a\alpha} (1 + \beta(t - t_0 + 1)) e^{2a\alpha} \Big\{ \Phi(1) e^{-a\alpha t/2} + \Phi(t/2+1) \Big\} \Big)$$

and Φ is defined as in the proof of Theorem 3.1. It is clear that $\Phi^*(t_0) \to 0$ as $t_0 \to$ since $t\,\Phi(t) \to 0$ as $t \to 0$. Hence, it follows from (3.5.7) that

$$E\,|\,X(t)\,|^2 \le c\,E\,|x_0|^2 + \Phi^*(t_0) + c \int_{t_0}^{t} \lambda(s)\, w(\,E\,|\,X(s)\,|^2\,)\, d\langle N\rangle_s$$

for all $t \ge t_0 \ge 1$. Therefore, by Lemma 3.2.3, we get

$$E\,|\,X(t)\,|^2 \le Q^{-1}\Big(Q(\,c\,E\,|x_0|^2 + \Phi^*(t_0)\,) + c \int_{t_0}^{t} \lambda(s)\, d\langle N\rangle_s \Big) \qquad (3.5.9)$$

where Q^{-1} is the inverse of Q, and Q is defined by

$$Q(v) = \int_{\varepsilon}^{v} \frac{dt}{w(t)}, \qquad v \ge 0, \;\; \varepsilon > 0$$

Now, we choose $E\,|x_0|^2$ and $\Phi^*(t_0)$ small enough so that $Q(\,c\,E\,|x_0|^2 + \Phi^*(t_0)\,)$ will be as large and negative as we desire (it will approach $-\infty$ arbitrarily). Then the right-hand side of (3.5.9) can be made arbitrarily small for all $t \ge t_0$. This prove that condition (1) of Definition 3.5.1 holds.

We shall further show condition (2) of Definition 3.5.1. Since condition (1) of Definition 3.5.1 holds, there exist positive constants δ_0, τ_0 and K such that

$$E\,|\,X(t, t_0, x_0)\,|^2 \le K \qquad \text{for all } t \ge t_0$$

whenever $E\,|x_0|^2 \le \delta_0$ and $t_0 \ge \tau_0$. Using (3.5.2), (3.5.3), (3.5.8) and the assumptions of the theorem, we then get

$$E\,|\,X(t)\,|^2 \qquad \le c\,E\,|x_0|^2 \exp(\,-a\alpha(\,t - t_0 - 1)\,)$$

$$+ \; c \int_0^{t/2} \exp(-a\alpha(t-s-1))\,\lambda(s)\,w(K)\,d\langle N\rangle_s$$

$$+ \; c \int_{t/2}^{t} \exp(-a\alpha(t-s-1))\,\lambda(s)\,w(K)\,d\langle N\rangle_s$$

$$+ \; (c + c\beta(t - t_o + 1)) \int_{t_o}^{t} \exp(-a\alpha\,(t-s-1))\,\gamma(s)\,d\langle N\rangle_s$$

$$\leq \; c\,E\,|x_o|^2 \exp(-a\alpha(t - t_o - 1))$$

$$+ \; c\,w(K) \exp(-a\alpha(t/2 - 1)) \int_0^{\infty} \lambda(s)\,d\langle N\rangle_s$$

$$+ \; c\,w(K)\,e^{a\alpha} \int_{t/2}^{\infty} \lambda(s)\,d\langle N\rangle_s$$

$$+ \; \frac{c}{a\alpha}\,(1 + \beta(t - t_o + 1))\,e^{2a\alpha} \left\{ \Phi(1)\,e^{-a\alpha t/2} + \Phi(t/2 + 1) \right\}$$

$$\to 0 \qquad \text{as } t \to \infty$$

which completes the proof.

3.6. EXPONENTIAL STABILITY

As we know, Lyapunov [1] in 1892 published his famous monograph "Probleme general de la stabilite du mouvement " in which he founded the theory of characteristic exponents that today bear his name. The original intention of Lyapunov was to determine criteria for the stability (of the origin x = 0) of

$$\dot{x} = A(t)\,x , \qquad x(0) = x_0 \in R^n, \quad t \in R_+ \tag{3.6.1}$$

and to conclude from the stability of Eq.(3.6.1) the stability for the nonlinear system

$$\dot{x} = A(t)\,x + f(t, x) , \qquad x(0) = x_0 \in R^n, \quad t \in R_+ \tag{3.6.2}$$

Hasminskii [2] (see also his book [3]) studied the stability of the origin of the linear stochastic differential equation

$$dx = Ax\,dt + \sum_{i=1}^{m} B_i\, x \circ dW_i , \qquad x(0) = x_0 \in R^n \tag{3.6.3}$$

which might be regarded as a stochastic perturbed system of Eq. (3.6.1). He gave a necessary and sufficient criterion for asymptotic stability of Eq.(3.6.3) which opened a new chapter in stochastic stability theory. Arnold, Oeljeklaus and Pardoux [1] studied more systematically the almost sure and moment stability for Eq. (3.6.3).

In this section we will consider the following problems. Given a stable stochastic linear system

$$X(t) = x_0 + \int_{t_0}^{t} A\, X(t)\, d\langle N\rangle_s , \qquad t \geq t_0\,(\geq 0) \tag{3.6.4}$$

where A is an n×n constant matrix, $N = (N_t)$ is a continuous local martingale with $N_0 = 0$ and $x_0 : \Omega \to R^n$ is $\mathcal{F}_{t_0}$-measurable with $E\,|\,x_0\,|^2 < \infty$. Assume that some parameters are excited or perturbed (environment noise), and the perturbed system has the form

$$X(t) = x_0 + \int_{t_0}^{t} A\, X(s)\, d\langle N\rangle_s + \int_{t_0}^{t} G(\,X(s), s\,)\, dN_s , \qquad t \geq t_0\,(\geq 0) \tag{3.6.5}$$

where $G : R^n \times R_+ \times \Omega \to R^n$ is sufficiently smooth for the existence and uniqueness of the solutions and $E\,|\,X(t, t_0, x_0)\,|^2 < \infty$ for all $t \geq t_0$ (cf. Chapter I), here $X(t, t_0, x_0)$

denotes the solution of Eq. (3.6.5). The question is: is Eq.(3.6.5) still stable? To answer this question we have the following theorem which gives sufficient conditions to show

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|X(t, t_0, x_0)|^2 \leq \text{const.} < 0$$

i.e., Eq.(3.6.5) is exponentially stable in mean square.

Theorem 3.6.1. Assume:

(1) There exist two positive constants c_1 and c_2 such that

$$|e^{At}|^2 \leq c_1 e^{-c_2 t}, \quad t \geq 0 \tag{3.6.6}$$

(which is equivalent to assume that all characteristic roots of A have negative real parts).

(2) For arbitrary $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$ such that

$$|\exp(-A\langle N\rangle_t)\,G(x, t)|^2 \leq \varepsilon\,|\exp(-A\langle N\rangle_t)\,x|^2 \quad \text{a.s.} \tag{3.6.7}$$

for all $x \in R^n$ and $t \geq T(\varepsilon)$.

(3) There exist positive constants α_i and β_i $(i = 1, 2)$ such that

$$\alpha_1 t - \beta_1 \leq \langle N\rangle_t \leq \alpha_2 t + \beta_2 \quad \text{a.s.} \quad t \geq 0 \tag{3.6.8}$$

Then

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|X(t, t_0, x_0)|^2 \leq -c_2\,\alpha_1 \tag{3.6.9}$$

for all $t_0 \geq 0$ and $\mathcal{F}_{t_0}$-measurable x_0 with $E\,|x_0|^2 < \infty$.

Proof. For arbitrary $\varepsilon > 0$, let $t_0 \geq T(\varepsilon)$. By Theorem 2.4.1, we have

$$X(\tau) = \exp\Big\{A\,(\langle N\rangle_\tau - \langle N\rangle_{t_0})\Big\}\,x_0 + \int_{t_0}^{\tau} \exp\Big\{A\,(\langle N\rangle_\tau - \langle N\rangle_s)\Big\}\,G(X(s), s)\,dN_s \tag{3.6.10}$$

for any stopping time τ with $t_0 \leq \tau < \infty$ a.s. It means

$$\exp(-A\langle N\rangle_\tau)\,X(\tau) = \exp(-A\langle N\rangle_{t_0})\,x_0 + \int_{t_0}^{\tau} \exp(-A\langle N\rangle_s)\,G(X(s), s)\,dN_s$$

Using Doob's martingale inequality (cf. Metivier [1], p. 55), we get

$$E\Big\{ \sup_{t_0 \le s \le \tau} | \exp(- A \langle N \rangle_s) X(s) |^2 \Big\}$$

$$\le \ c_3 + 8\, E \int_{t_0}^{\tau} | \exp(- A \langle N \rangle_s) G(X(s), s) |^2 \, d\langle N \rangle_s$$

$$\le \ c_3 + 8\, \varepsilon\, E \int_{t_0}^{\tau} | \exp(- A \langle N \rangle_s) X(s) |^2 \, d\langle N \rangle_s$$

$$\le \ c_3 + 8\varepsilon\, E \int_{t_0}^{\tau} \sup_{t_0 \le u \le s} | \exp(-A \langle N \rangle_u) X(u) |^2 \, d\langle N \rangle_s \qquad (3.6.11)$$

where $c_3 = 2\, E | x_0 \exp(-A \langle N \rangle_{t_0}) |^2$. Applying Lemma 3.2.6 to (3.6.11), we get

$$E\Big\{ \sup_{t_0 \le u \le t} | \exp(-A \langle N \rangle_u) X(u) |^2 \Big\} \le c_4\, e^{8\varepsilon\alpha_2 t}, \quad t \ge t_0 \qquad (3.6.12)$$

where $c_4 = c_3 e^{8\varepsilon\beta_2}$. Hence,

$$\begin{aligned} E | X(t) |^2 &\le E \Big\{ | \exp(A \langle N \rangle_t) |^2 \, | \exp(-A \langle N \rangle_t) X(t) |^2 \Big\} \\ &\le E \Big\{ c_1 \exp(- c_2 \langle N \rangle_t) \, | \exp(-A \langle N \rangle_t) X(t) |^2 \Big\} \\ &\le c_1 \exp\{ - c_2 (\alpha_1 t - \beta_1) \} \, E | \exp(-A \langle N \rangle_t) X(t) |^2 \\ &\le c_1 \exp\{ - c_2 (\alpha_1 t - \beta_1) \} \, c_4 \exp(8\varepsilon\alpha_2 t) \\ &\le c_5 \exp\{ - (c_2\alpha_1 - 8\varepsilon\alpha_2) t \} \end{aligned}$$

where $c_5 = c_1 c_4 e^{c_2 \beta_1}$. It follows immediately that

$$\overline{\lim_{t \to \infty}} \ \frac{1}{t} \log E | X(t, t_0, x_0) |^2 \le - c_2\alpha_1 + 8\varepsilon\alpha_2 \qquad (3.6.13)$$

provided $t_0 \ge T(\varepsilon)$ and x_0 is $\mathcal{F}_{t_0}$-measurable with $E | x_0 |^2 < \infty$. Finally, for

$0 \le t_o < T(\varepsilon)$, we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,X(t, t_o, x_o)\,|^2$$

$$= \overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,X(t, T(\varepsilon), X(T(\varepsilon), t_o, x_o)\,|^2 \le -c_2\alpha_1 + 8\varepsilon\alpha_2$$

This means

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,X(t, t_o, x_o)\,|^2 \le -c_2\alpha_1 + 8\varepsilon\alpha_2$$

for all $t_o \ge 0$ and $\mathcal{F}_{t_o}$-measurable x_o with $E\,|\,x_o\,|^2 < \infty$. Letting $\varepsilon \to 0$, we get

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,X(t, t_o, x_o)\,|^2 \le -c_2\alpha_1$$

which is just what we want to prove.

Let us now consider the more general SIES

$$Y(t) = x_o + \int_{t_o}^{t} F(Y(s), s)\, d\langle M\rangle_s + \int_{t_o}^{t} G(Y(s), s)\, dM_s$$

$$+ \int_{t_o}^{t} f(Y(s), s)\, d\langle N\rangle_s + \int_{t_o}^{t} g(Y(s), s)\, dN_s, \quad t \ge t_o\ (\ge 0) \qquad (3.6.14)$$

where M, like N, is a continuous local martingale with $M_0 = 0$ a.s. and F, G, f, g : $R^n \times R_+ \times \Omega \to R^n$ are sufficiently smooth for the existence and uniqueness of the solutions (cf. Chapter I). Eq. (3.6.14) may regarded as a perturbed system of the followingequation

$$X(t) = x_o + \int_{t_o}^{t} F(X(s), s)\, d\langle M\rangle_s + \int_{t_o}^{t} G(X(s), s)\, dM_s \qquad (3.6.15)$$

Let $Y(t) = Y(t, t_o, x_o)$ and $X(t) = X(t, t_o, x_o)$ be the solutions of Eq. (3.6.14) and (3.6.15), respectively. A natural question arises: under what conditions are the stability properties of Eq. (3.6.15) shared by the solutions of Eq. (3.6.14)? For this we have the following theorem.

Theorem 3.6.2. Suppose the solution of Eq. (3.6.15) satisfies

$$E|X(t,t_0,x_0)|^2 \le c_1 E|x_0|^2 e^{-c_2(t-t_0)}, \quad t \ge t_0 \tag{3.6.16}$$

for all sufficiently large t_0 and $\mathcal{F}_{t_0}$-measurable x_0 with $E|x_0|^2 < \infty$, where c_1 and c_2 are positive constants. Furthermore, assume that for some functions $A, B : (0,\infty) \to (0,\infty)$:

(1) $$\sup_{t\ge 0} (\langle N\rangle_{t+\tau} - \langle N\rangle_t) \le A(\tau) \quad \text{a.s.} \quad \tau > 0 \tag{3.6.17}$$

(3) $$|F(x,t) - F(y,t)|^2 \vee |G(x,t) - G(y,t)|^2 \le L(t)|x-y|^2 \quad \text{a.s.} \tag{3.6.18}$$

for all $x, y \in R^n$, $t \ge 0$, where $L(t)$ is a nonnegative progressive process satisfying

$$\sup_{t\ge 0}\left\{(\langle M\rangle_{t+\tau} - \langle M\rangle_t + 1)\int_t^{t+\tau} L(s)\,d\langle M\rangle_s\right\} \le B(\tau) \quad \text{a.s.} \quad \tau > 0 \tag{3.6.19}$$

(4) $$|f(x,t)|^2 \vee |g(x,t)|^2 \le \gamma(t) \quad \text{a.s.} \quad x \in R^n,\ t \ge 0 \tag{3.6.20}$$

where $\gamma(t)$ is a nonnegative progressive process satisfying

$$Q(t_0) := \sup\{\Gamma(s):\ t_0 - 1 \le s < \infty\} \le c_3 e^{-c_4 t_0} \tag{3.6.21}$$

for all sufficiently large t_0, here c_3, c_4 are positive constants and

$$\Gamma(t) := E\int_t^{t+1} \gamma(s)\,d\langle N\rangle_s \tag{3.6.22}$$

Then, for the solutions of Eq. (1.10), we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log E|Y(t,t_0,x_0)|^2 \le -(c_2 \wedge c_4) \tag{3.6.23}$$

for all $t_0 \ge 0$ and $\mathcal{F}_{t_0}$-measurable x_0 with $E|x_0|^2 < \infty$.

Proof. Step 1. We first prove

$$E\left\{\sup_{t_0\le s\le t_0+\tau} |Y(s) - X(s)|^2\right\} \le 4(\tau+1)(A(\tau)+4)\,Q(t_0)\,e^{16B(\tau)} \tag{3.6.24}$$

if $t_0 \ge 1$, $E|x_0|^2 < \infty$ and $\tau > 0$. Indeed, for any stopping time σ with $t_0 \le \sigma$

$\leq t_o + \tau$, we have

$$E\left\{ \sup_{t_o \leq s \leq \sigma} |Y(s) - X(s)|^2 \right\}$$

$$\leq 4E\left\{ (\langle M\rangle_\sigma - \langle M\rangle_{t_o} + 4) \int_{t_o}^{\sigma} L(s)\,|Y(s) - X(s)|^2\, d\langle M\rangle_s \right\}$$

$$+ 4E\left\{ (\langle N\rangle_\sigma - \langle N\rangle_{t_o} + 4) \int_{t_o}^{\sigma} \gamma(s)\, d\langle N\rangle_s \right\}$$

where Cauchy's inequality and Doob's martingale inequality have been used. Hence

$$E\left\{ \sup_{t_o \leq s \leq \sigma} |Y(s) - X(s)|^2 \right\}$$

$$\leq 16E\left\{ (\langle M\rangle_{t_o+\tau} - \langle M\rangle_{t_o} + 1) \int_{t_o}^{\sigma} L(v)\,\{ \sup_{t_o \leq s \leq v} |Y(s) - X(s)|^2 \}\, d\langle M\rangle_v \right\}$$

$$+ 4(A(\tau) + 1)E \int_{t_o}^{t_o+\tau} \gamma(s)\, d\langle N\rangle_s$$

Applying Lemma 3.2.6 to this, we get

$$E\left\{ \sup_{t_o \leq s \leq t_o+\tau} |Y(s) - X(s)|^2 \right\} \leq 4(A(\tau) + 1)\, e^{16B(\tau)}\, E \int_{t_o}^{t_o+\tau} \gamma(s)\, d\langle N\rangle_s \quad (3.6.25)$$

However, we have

$$\int_{t_o-1}^{t_o+\tau} \Gamma(s)\, ds = E \int_{t_o-1}^{t_o+\tau} \left(\int_{s}^{s+1} \gamma(r)\, d\langle N\rangle_s \right) ds \geq E \int_{t_o}^{t_o+\tau} \gamma(s)\, d\langle N\rangle_s$$

by changing the order of integration. Hence, it follows from (3.6.25) that

$$E\left\{\sup_{t_o\le s\le t_o+\tau} |Y(s) - X(s)|^2\right\} \le 4\,(A(\tau)+1)\,e^{16B(\tau)} \int_{t_o-1}^{t_o+\tau} \Gamma(s)\,ds$$

$$\le 4\,(\tau+1)\,(A(\tau)+1)\,Q(t_o)\,e^{16B(\tau)}$$

which is (3.6.24).

Step 2. For arbitrary $\varepsilon > 0$, choose T so large that conditions (3.6.16) and (3.6.21) hold for all $t_o \ge T$ and that $\log(2c_1)/T < \varepsilon$. Set $\theta = c_2 \wedge c_4$. From (3.6.24) and (3.6.21) we obtain

$$E\left\{\sup_{nt_o\le t\le(n+1)t_o} |Y(t, nt_o, x_o) - X(t, nt_o, x_o)|^2\right\} \le c_5\, e^{-\theta n t_o} \tag{3.6.26}$$

for $n = 1, 2, \ldots,$ provided x_o is $\mathcal{F}_{nt_o}$-measurable with $E|x_o|^2 < \infty$, where

$$c_5 = 4\,(t_o+1)\,(A(t_o)+4)\,e^{16B(t_o)}$$

Step 3. Now, let $t_o \ge T$ and x_o is $\mathcal{F}_{t_o}$-measurable with $E|x_o|^2 < \infty$. Without loss of any generality, we can assume $c_1 \ge 1$. We now prove, by induction, that

$$E|Y(t, t_o, x_o)|^2 \le 2^n c_1^n\,(n c_5 + E|x_o|^2)\,e^{-\theta(t-t_o)} \tag{3.6.27}$$

for $t \in [nt_o, (n+1)t_o]$, $n = 1, 2, \cdots$. Indeed, for $t \in [t_o, 2t_o]$, we get from (3.6.26) and (3.6.16) that

$$E|Y(t, t_o, x_o)|^2 \le 2E|Y(t, t_o, x_o) - X(t, t_o, x_o)|^2 + 2E|X(t, t_o, x_o)|^2$$

$$\le 2c_5 e^{-\theta t_o} + 2c_1 E|x_o|^2 e^{-\theta(t-t_o)}$$

$$\le 2c_1 (c_5 + E|x_o|^2)\,e^{-\theta(t-t_o)}$$

which means (3.6.27) holds for $n = 1$. Assume (3.6.27) holds for $n = m$. Then

$$E|Y_m|^2 \le 2^m c_1^m\,(m c_5 + E|x_o|^2)\,e^{-\theta m t_o}$$

where $Y_m := Y((m+1)t_o, t_o, x_o)$. Therefore, for $t \in [(m+1)t_o, (m+2)t_o]$, we have

$$E|Y(t, t_o, x_o)|^2$$

$$\le 2E|Y(t, (m+1)t_o, Y_m) - X(t, (m+1)t_o, Y_m)|^2 + 2E|X(t, (m+1)t_o, Y_m)|^2$$

$$\leq \ 2\, c_5\, e^{-\theta(m+1)t_0} + (2\, c_1)^{m+1} (m\, c_5 + E\,|\, x_0\,|^2)\, e^{-\theta m t_0 - \theta[\, t-(m+1)t_0]}$$

$$\leq \ (2\, c_1)^{m+1} \{ (m+1)\, c_5 + E\,|\, x_0\,|^2 \}\, e^{-\theta(t-t_0)}$$

(3.6.27) has been proved.

Step 4. It follows from (3.6.27) that

$$\overline{\lim_{t\to\infty}} \ \frac{1}{t} \log E\,|\, Y(t, t_0, x_0)\,|^2$$

$$\leq \ \overline{\lim_{n\to\infty}} \ \frac{1}{nt_0} \log \left\{ 2^n c_1{}^n (n\, c_5 + E\,|\, x_0\,|^2)\, e^{-\theta(n-1)t_0} \right\}$$

$$\leq \ \log(2c_1)\,/\,t_0 - \theta \ \leq \ \varepsilon - \theta \qquad (3.6.28)$$

provided $t_0 \geq T$ and x_0 is $\mathcal{F}_{t_0}$-measurable with $E\,|\, x_0\,|^2 < \infty$. For $t_0 \in [0, T)$ and $\mathcal{F}_{t_0}$-measurable x_0 with $E\,|\, x_0\,|^2 < \infty$, we have

$$\overline{\lim_{t\to\infty}} \ \frac{1}{t} \log E\,|\, Y(t, t_0, x_0)\,|^2$$

$$\leq \ \overline{\lim_{t\to\infty}} \ \frac{1}{t} \log E\,|\, Y(t, T, Y(T, t_0, x_0))\,|^2 \ \leq \ \varepsilon - \theta \qquad (3.6.29)$$

Letting $\varepsilon \to 0$ in (3.6.28) and (3.6.29), we get stated result (3.6.23). The proof is cpmplete.

In Theorem 3.6.2, condition (1) means that Eq.(3.6.15) is exponentially stable. Naturally, we may ask how to check this condition. Using the Lyapunov function, we will prove the following theorem.

Theorem 3.6.3. Assume:

(1) There exist two constants $b > 0$ and $c \geq 0$ such that

$$b\,(t - s) - c \ \leq \ \langle M\rangle_t - \langle M\rangle_s \quad \text{a.s.} \quad 0 \leq s \leq t < \infty \qquad (3.6.30)$$

(2) There exists a function $V(x, t) : R^n{\times}R_+ \to R_+$ with continuous second order partial derivatives in x and first order partial derivative in t such that

$$\alpha\,|\, x\,|^2 \ \leq \ V(x, t) \ \leq \ \beta\,|\, x\,|^2, \qquad (x, t) \in R^n{\times}R_+ \qquad (3.6.31)$$

and

$$L\,V(x,\ \langle M\rangle_t - \langle M\rangle_s) := V_t(x,\ \langle M\rangle_t - \langle M\rangle_s) +$$

$$+\ V_x(x,\ \langle M\rangle_t - \langle M\rangle_s)\cdot G(x,t) + \frac{1}{2}G(x,t)^T\cdot V_{xx}(x, \langle M\rangle_t - \langle M\rangle_s)\cdot G(x,t)$$

$$\le -\gamma\,|x|^2 \quad \text{a.s.} \quad x\in R^n,\quad 0\le s\le t<\infty \tag{3.6.32}$$

where $\alpha,\ \beta,\ \gamma$ are positive constants.

Then the solutions of Eq.(3.6.15) satisfy

$$E\,|X(t,t_o,x_o)|^2 \le \frac{\alpha}{\beta}E\,|x_o|^2\exp\Big\{\frac{\gamma}{\beta}[\,c-b(t-t_o)\,]\Big\},\quad t\ge t_o \tag{3.6.33}$$

for all $t_o\ge 0$ and and $\mathcal{F}_{t_o}$-measurable x_o with $E\,|x_o|^2<\infty$.

Proof. Fix t_o and x_o. Set

$$x(t) = X(t_o+t,\ t_o,\ x_o), \qquad m(t) = M_{t_o+t} - M_{t_o}$$

$$h(x,t) = F(x,t_o+t), \qquad k(x,t) = G(x,t_o+t), \qquad \mathcal{G}_t = \mathcal{F}_{t_o+t}$$

for $0\le t<\infty$. Then x(t) is the solution of the equation

$$dx(t) = x_o + \int_0^t h(x(s),s)\,d\langle m\rangle_s + \int_0^t g(x(s),s)\,dm_s,\quad t\ge 0 \tag{3.6.34}$$

where the stochastic integral is relative to $\{\mathcal{G}_t\}_{t\ge 0}$. By Itô's formula and assumptions (3.6.31) and (3.6.32), we deduce

$$E\,V(x(\tau),\langle m\rangle_\tau) - E\,V(x(\sigma),\langle m\rangle_\sigma) = E\int_\sigma^\tau L\,V(x(s),\ \langle m\rangle_s)\,d\langle m\rangle_s$$

$$\le -\frac{\gamma}{\beta}E\int_\sigma^\tau V(x(s),\ \langle m\rangle_s)\,d\langle m\rangle_s \tag{3.6.35}$$

for any $\{\mathcal{G}_t\}$-stopping times σ and τ with $0\le\sigma\le\tau<\infty$ a.s. Define

$$T_t := \inf\{\,s\ge 0:\ \langle m\rangle_s\ge t\,\},\quad t\ge 0$$

It is easy to check that T_t is a finite $\{\mathcal{G}_t\}$-stopping time and $\langle m\rangle_{T_t} = t$. Hence, from (3.6.35) we get

$$E\,V(x(T_t),t) - E\,V(x(T_s),s) \le -\frac{\gamma}{\beta} E \int_{T_s}^{T_t} V(x(u), \langle m\rangle_u)\, d\langle m\rangle_u$$

$$\le -\frac{\gamma}{\beta} E \int_s^t V(x(T_u),u)\,du \le -\frac{\gamma}{\beta} \int_s^t E\,V(x(T_u),u)\,du \qquad (3.6.36)$$

for all $0 \le s \le t < \infty$, here Lemma 3.2.1 has been used. An application of Lemma 3.2.4 to (3.6.36) yields

$$E\,V(x(T_t),t) \le E\,V(x(0),0)\exp(-\frac{\gamma}{\beta}t)$$

$$\le \beta E|x_0|^2 \exp(-\frac{\gamma}{\beta}t), \qquad t \ge 0 \qquad (3.6.37)$$

Now, for any $t \ge 0$, we have $T_{(bt-c)\vee 0} \le t$ by assumption (3.6.30). Thus, it follows from (3.6.35) and (3.6.37) that

$$E\,V(x(t),\langle m\rangle_t) \le E\,V(x(T_{(bt-c)\vee 0}), (bt-c)\vee 0)$$

$$\le \beta E|x_0|^2 \exp\left\{-\frac{\gamma}{\beta}\{(bt-c)\vee 0\}\right\} \le \beta E|x_0|^2 \exp\left\{-\frac{\gamma}{\beta}(bt-c)\right\}$$

which implies immediately that

$$E|X(t,t_0,x_0)|^2 \le \frac{\alpha}{\beta} E|x_0|^2 \exp\left\{\frac{\gamma}{\beta}[c - b(t-t_0)]\right\}, \qquad t \ge t_0$$

This is the stated result (3.6.33), and the proof is complete.

We now give some examples to illustrate our results.

Example 3.6.4. Let us first consider the following one-dimensional linear SIES

$$X(t) = x_0 - \int_{t_0}^t a X(s)\, d\langle N\rangle_s + \int_{t_0}^t b(t)\sin(X(t))\,dN_s, \quad t \ge t_0\ (\ge 0) \qquad (3.6.38)$$

where a is a positive constant and $b(t): R_+ \to R$ such that $b(t) \to 0$ as $t \to \infty$, and N is a continuous local martingale. Suppose there exist positive constants α_i and β_i ($i = 1, 2$) such that

$$\alpha_1 t - \beta_1 \leq \langle N \rangle_t \leq \alpha_2 t + \beta_2 \quad \text{a.s.} \quad \text{for all } t \geq 0$$

Applying Theorem 3.6.1 to this equation, we deduce that the solutions satisfy

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|X(t, t_0, x_0)|^2 \leq -2a\alpha_1$$

In particular, if N is a Wiener process, we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|X(t, t_0, x_0)|^2 \leq -2a$$

Example 3.6.5. We now consider the two-dimensional SIES

$$X(t) = x_0 + \int_{t_0}^{t} F(X(s), s)\, d\langle M \rangle_s + \int_{t_0}^{t} G(X(s), s)\, dM_s, \quad t \geq t_0 (\geq 0) \tag{3.6.39}$$

where

$$F(x, t) = \begin{bmatrix} -x_1 + t^2 x_2 - t x_1^3 \\ -x_2 - t^2 x_1 - t x_2^3 \end{bmatrix} \qquad G(x,t) = \begin{bmatrix} -(2t)^{1/2} x_1^2 \\ (2t)^{1/2} x_2^2 \end{bmatrix} \tag{3.6.40}$$

and M is a continuous local martingale such that there exist two constants $b > 0$ and $c \geq 0$ such that

$$b(t-s) - c \leq \langle M \rangle_t - \langle M \rangle_s \quad \text{a.s.} \quad 0 \leq s \leq t < \infty$$

Set the Lyapunov function

$$V(x, t) = x_1^2 + x_2^2$$

Then we have

$$\begin{aligned} & L\,V(x, \langle M \rangle_t - \langle M \rangle_s) \\ &= -2x_1(x_1 - t^2 x_2 + t x_1^3) - 2x_2(x_2 + t^2 x_1 + t x_2^3) \\ &\qquad + 2t(x_1^4 + x_2^4) \\ &= -2(x_1^2 + x_2^2) \end{aligned}$$

Hence, by Theorem 3.6.3, we have

$$E\,|X(t, t_0, x_0)|^2 \leq E\,|x_0|^2\, e^{2[c - b(t - t_0)]}, \quad t \geq t_0$$

Example 3.6.6. Suppose Eq.(3.6.39) is perturbed by environment noise, and the perturbed system has the form

$$Y(t) = x_0 + \int_{t_0}^{t} F(Y(s), s)\, d\langle M\rangle_s + \int_{t_0}^{t} G(Y(s), s)\, dM_s$$

$$+ \int_{t_0}^{t} f(Y(s), s)\, ds + \int_{t_0}^{t} g(Y(s), s)\, dW_s, \quad t \geq t_0 \ (\geq 0) \tag{3.6.41}$$

where W is a Wiener process, F and G are defined by (3.6.40) and

$$f(x, t) = e^{-t} \begin{bmatrix} -\sin(x_1^2) \\ \cos(x_2) \end{bmatrix} \qquad g(x, t) = e^{-2t} \begin{bmatrix} \cos(x_2) \\ -\sin(x_1^2) \end{bmatrix}$$

Application of Theorem 3.6.2 to this perturbed system, together with the conclusion of Example 3.6.5, implies that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,Y(t, t_0, x_0)\,|^2 \leq -(2b \wedge 2)$$

Chapter IV

Comparison Theorems

4.1. INTRODUCTION

In the investigation of the solutions of stochastic integral equations, comparison theorems are very powerful tools as in the case for deterministic ones. Many authors like Anderson [1], Ikeda and Watanabe [1], Skorokhod [1], O'Brien [1], Gal'cuk and Davis [1] have given several comparison theorems for the solutions of stochastic differential equations. In this chapter we first study the comparison theorems between a stochastic integral equation and a deterministic differential equation which are mainly based on our paper published in 1986. We then give several comparison theorems for the solutions of two stochastic integral equations with respect to semimartingales. Several examples are worked out to illustrate our results.

4.2. COMPARISON THEOREMS BETWEEN SIES AND DETERMINISTIC DIFFERENTIAL EQUATIONS

Consider the following SIES

$$X(t) = x_0 + \int_0^t F(X)(s)\,ds + \sum_{i=1}^{m} \int_0^t F_i(X)(s)\,dM_i(s), \qquad t \geq 0 \qquad (4.2.1)$$

where $x_0 \in R^n$, F and F_i $(1 \leq i \leq m)$: $\mathcal{X} \to \mathcal{P}$ satisfy the conditions for the existence and uniqueness of the solutions discussed in Chapter I, M_i $(1 \leq i \leq m)$ are continuous semimartingales such that $M_i(0) = 0$ and

$$\langle M_i^c, M_j^c \rangle(t) = \int_0^t m_{ij}(s)\,ds, \qquad t \geq 0, \quad 1 \leq i, j \leq m \qquad (4.2.2)$$

in which m_{ij} are all integrable adapted processes. Denote by $X(t, x_0)$ the solution of Eq.(4.2.1).

We will also consider the following deterministic differential equation

$$\begin{cases} \dfrac{d}{dt} r(t) = W(t, f^{-1}(r(t))), & t_o \le t < \infty \\ r(t_o) = r_o & (t_o \ge 0) \end{cases} \tag{4.2.3}$$

where $r_o \in R_+$, $f \in C^2(R_+, R_+)$ is strictly increasing and convex, $W \in C(R_+ \times R_+, R_+)$ and $W(t, u)$ is nondecreasing and concave in u for each t. We assume there exists a maximal solution, say $r(t, t_o, r_o)$, of Eq.(4.2.3) existing on R_+.

We will need some notations. Define the operator

$$\mathcal{L}U(X(t), t) = U_x(X(t), t)\, F(X)(t)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{m} m_{ij}(t)\, (F_i(X)(t))'\, U_{xx}(X(t), t)\, F_j(X)(t)$$

Let $X \in \mathcal{X}$ and $h > 0$. Define

$$\tau_h = \tau_h(X) = \inf\{ t \ge 0 : X(t) \notin S_h \}$$

and set $X^h = X^{\tau_h}$. We also need the following assumptions.

(A1) There exists a function $V \in C^{2,1}(\underline{S}_h \times R_+)$ such that

(1) for any $x_o \in S_h$ and $t \ge 0$

$$\left\{ (f \circ V)_t(X^h(t, x_o), t) + \mathcal{L}\, (f \circ V)(X^h(t, x_o), t) \right\} \mathbb{1}_{]\!]0, \tau_h]\!]}(t)$$

$$\le W(t, V(X^h(t, x_o), t))\, \mathbb{1}_{]\!]0, \tau_h]\!]}(t)$$

and

$$W(t, V(X^h(t, x_o), t)) - (f \circ V)_t(X^h(t, x_o), t) \ge 0$$

where $(f \circ V) = f(V)$;

(2) for any $x_o \in S_h$, $t_o \ge 0$ and finite stopping time T with $t_o \le T$ a.s.

$$E\left\{ \sum_{i=1}^{m} \int_{t_o}^{T} (f \circ V)_x(X^h(s, x_o), s)\, F_i(X^h(x_o))(s)\, dM_i(s) \right\} \le 0$$

provided $\int_{t_o}^{T} (f \circ V)_x(X^h(s, x_o), s)\, F_i(X^h(x_o))(s)\, dM_i(s)$ $(1 \le i \le m)$ are integrable.

(A2) There exists a function $V \in C^{2,1}(R^n \times R_+)$ such that

(1) for any $x_o \in R^n$ and $t \ge 0$

$$(f \circ V)_t(X(t, x_o), t) + \mathcal{L}(f \circ V)(X(t, x_o), t) \le W(t, V(X(t, x_o), t)) \le K(t)$$

where K is a nonnegative integrable adapted process such that

$$E\int_0^t K(s)\, ds < \infty \qquad \text{for all } t \ge 0$$

(2) for any $x_o \in R^n$, $t_o \ge 0$ and finite stopping time T with $t_o \le T$ a.s.

$$E\left\{ \sum_{i=1}^{m} \int_{t_o}^{T} (f \circ V)_x(X(s, x_o), s)\, F_i(X(x_o))(s)\, dM_i(s) \right\} \le 0$$

provided $\int_{t_o}^{T} (f \circ V)_x(X(s, x_o), s)\, F_i(X(x_o))(s)\, dM_i(s)$ $(1 \le i \le m)$ are integrable.

We now prove the following comparison theorems.

Theorem 4.2.1. Assume (A1) holds. Then for any $x_o \in S_h$, $t_o \ge 0$ and $0 < \rho \le h$, we have

$$E\, V(X^h(\sigma \wedge t, x_o), \sigma \wedge t) \le f^{-1}\Big[r\Big(t, t_o, E\, f(V(X^h(t_o, x_o), t_o))\Big)\Big] \qquad (4.2.4)$$

for all $t \ge t_o$, where

$$\sigma = \sigma(\rho, t_o) = \inf\{t \ge t_o : X^h(t, x_o) \notin \underline{S}_\rho\}$$

Particularly, we have

$$E\, V(X^h(t, x_o), t) \le f^{-1}\Big[r\Big(t, t_o, E\, f(V(X^h(t_o, x_o), t_o))\Big)\Big], \quad t \ge t_o \qquad (4.2.5)$$

Proof. Since (4.2.5) follows from (4.2.4) by taking $\rho = h$, it is enough to prove (4.2.4). It follows from (4.2.1) that

$$X^h(t) = x_o + \int_0^t F(X^h)(s)\, 1\!\!1_{]\!]0,\tau_h]\!]}(s)\, ds + \sum_{i=1}^m \int_0^t F_i(X^h)(s)\, 1\!\!1_{]\!]0,\tau_h]\!]}(t)\, dM_i(s)$$

Define stopping times

$$T_k = \inf\{ t \geq t_o : \Big| \sum_{i=1}^m \int_{t_o}^t (f \circ V)_x(X^h(s, x_o), s)\, F_i(X^h(x_o))(s)\, dM_i(s) \Big| \geq k \}$$

for $k = 1, 2, \ldots$. It is clear that $T_k \uparrow \infty$ a.s. By Itô's formula and assumption (A1), we deduce

$$E\, f\big[V(X^h(\sigma\wedge t\wedge T_k, x_o), \sigma\wedge t\wedge T_k) \big] \leq E\, f\big[V(X^h(t_o, x_o), t_o) \big]$$

$$+ E \int_{t_o}^{\sigma\wedge t\wedge T_k} \Big\{ (f\circ V)_t(X^h(s, x_o), s) + \mathfrak{L}\, (f\circ V)(X^h(s, x_o), s)\, 1\!\!1_{]\!]0,\tau_h]\!]}(s) \Big\}\, ds$$

$$\leq E\, f\big[V(X^h(t_o, x_o), t_o) \big] + E \int_{t_o}^{\sigma\wedge t\wedge T_k} W(s, V(X^h(s, x_o), s))\, ds$$

It follows by letting $k \to \infty$ and using Fatou's lemma that

$$E\, f\big[V(X^h(\sigma\wedge t, x_o), \sigma\wedge t) \big]$$

$$\leq E\, f\big[V(X^h(t_o, x_o), t_o) \big] + E \int_{t_o}^{\sigma\wedge t} W(s, V(X^h(s, x_o), s))\, ds$$

$$\leq E\, f\big[V(X^h(t_o, x_o), t_o) \big] + E \int_{t_o}^{t} W(s, V(X^h(\sigma\wedge s, x_o), \sigma\wedge s))\, ds$$

Applying Jensen's inequality and Fubini's theorem to this we get

$$f\big[EV(X^h(\sigma\wedge t, x_o), \sigma\wedge t) \big] \leq$$

$$\leq \; E f\left[V(X^h(t_0, x_0), t_0)\right] + \int_{t_0}^{t} W(s, EV(X^h(\sigma \wedge s, x_0), \sigma \wedge s) \;\; ds$$

which implies desired inequality (4.2.4) by using Theorem 3.1 of Lak   mikantham [1].
The proof is complete.

Theorem 4.2.2. Assume (A2) holds. Then for any $x_0 \in R^n$, ≥ 0 and finite stopping time ζ with $t_0 \leq \zeta$ a.s., we have

$$E\, V(X(\zeta \wedge t, x_0), \zeta \wedge t) \;\leq\; f^{-1}\left[r\left(t, t_0, E\, f(V(X(t_0, x_0), t_0) \right) \right) \tag{4.2.6}$$

for all $t \geq t_0$.

Proof. Define stopping times

$$T_k \;=\; \inf\{\, t \geq t_0 : \left| \sum_{i=1}^{m} \int_{t_0}^{t} (f \circ V)_x(X(s, x_0), s)\, F_i(X(x_0))(s)\, dM_i(s) \right| \geq k \,\}$$

for $k = 1, 2, \ldots$. By Itô's formula we have

$$f\left[V(X(\zeta \wedge t \wedge T_k, x_0), \zeta \wedge t \wedge T_k)\right] \;=\; f\left[V(X(t_0, x_0), t_0)\right] \;+$$

$$+ \int_{t_0}^{\zeta \wedge t \wedge T_k} \left\{ (f \circ V)_t(X(s, x_0), s) \;+\; \mathfrak{L}(f \circ V)(X(s, x_0), s) \right\} ds$$

$$+ \sum_{i=1}^{m} \int_{t_0}^{\zeta \wedge t \wedge T_k} (f \circ V)_x(X(s, x_0), s)\, F_i(X(x_0))(s)\, dM_i(s) \tag{4.2.7}$$

for all $x_0 \in R^n$, $0 \leq t_0 \leq t < \infty$ and $k = 1, 2, \ldots$. If we let $t_0 = ($ we can use assumption (A2) to deduce from (4.2.7) that

$$E\, f\left[V(X(\zeta \wedge t \wedge T_k, x_0), \zeta \wedge t \wedge T_k)\right]$$

$$\leq \; f\left[V(x_0, 0)\right] \;+\; E \int_0^t K(s)\, ds$$

which implies

$$E f\left[V(X(\zeta \wedge t, x_0), \zeta \wedge t)\right] < \infty$$

immediately. Particularly,

$$E f\left[V(X(t_0, x_0), t_0)\right] < \infty$$

Therefore, it follows from (4.2.7) that

$$E f\left[V(X(\zeta \wedge t \wedge T_k, x_0), \zeta \wedge t \wedge T_k)\right]$$

$$\leq E f\left[V(X(t_0, x_0), t_0)\right] + E \int_{t_0}^{\zeta \wedge t \wedge T_k} W(s, V(X(s, x_0), s))\Big\} ds$$

Letting $k \to \infty$ and then using Fubini's theorem and Jensen's inequality we can get

$$f\left[EV(X(\zeta \wedge t, x_0), \zeta \wedge t)\right]$$

$$\leq E f\left[V(X(t_0, x_0), t_0)\right] + \int_{t_0}^{t} W(s, EV(X(\zeta \wedge s, x_0), \zeta \wedge s))\Big\} ds$$

which implies desired result (4.2.6). We complete the proof.

There are many applications of the comparison theorems. For instance, we can employ Theorem 4.2.2 to study the stochastic boundedness (cf. Section 3.4). In fact, we have the following theorem.

Theorem 4.2.3. Assume

(1) (A2) holds;

(2) $\lim_{|x| \to \infty} V(x, t) = \infty$ uniformly in $t \geq 0$;

(3) for any $\delta > 0$ there exists a positive number $B(\delta)$ such that

$$r(t, 0, r_0) < B, \qquad t \geq 0$$

provided $0 \leq r_0 \leq \delta$.

Then the solutions of Eq.(4.2.1) are stochastically bounded.

Proof. Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary. It follows from assumption (3) there exists a positive number $B_1(\delta)$ such that for $x_0 \in S_\delta$,

$$f^{-1}\left[r(t, 0, f(x_0))\right] < B_1, \qquad t \geq 0 \tag{4.2.8}$$

By condition (2) there also exists a positive number $B_2(\varepsilon, \delta)$ such that

$$V(x,t) \geq \frac{B_1}{\varepsilon}, \qquad (x,t) \in (R^n - S_{B_2}) \times R_+ \tag{4.:}$$

For any $x_0 \in S_\delta$, we now define stopping time

$$\zeta = \inf\{ t \geq 0 : X(t, x_0) \notin S_{B_2} \}$$

By Theorem 4.2.2 and inequalities (4.2.8) and (4.2.9) we have

$$B_1 \geq E\, V(X(\zeta \wedge t, x_0), \zeta \wedge t)$$

$$\geq E\Big\{ V(X(\zeta, x_0), \zeta)\, \mathbb{1}_{\{\zeta \leq t\}} \Big\} \geq \frac{B_1}{\varepsilon} P\{ \zeta \leq t \}$$

for all $t \geq 0$, which implies

$$P\{ \zeta < \infty \} \leq \varepsilon$$

Namely

$$P\{ X(t, x_0) \in S_{B_2} \} \geq 1 - \varepsilon$$

The proof is complete.

Similarly, we can use Theorem 4.2.1 to prove the following theorem about stochastic stability (cf. Section 3.3.)

Theorem 4.2.4. Assume

(1) (A1) holds;

(2) $F(0) = F_i(0) = 0$ $(1 \leq i \leq m)$, $f(0) = 0$, $W(t, 0) = 0$ for all $t \geq 0$;

(3) $\mu(|x|) \leq V(x,t)$ for all $(x,t) \in S_h \times R_+$, where $\mu : R_+ \to R_+$ such $\mu(\alpha)$: for $\alpha > 0$;

(4) for any $\rho > 0$ there exists a positive number $\delta(\rho)$ such that

$$r(t, 0, r_0) < \rho, \quad t \geq 0$$

provided $0 \leq r_0 < \delta$.

Then the zero solution of Eq.(4.2.1) is stochastic stable.

4.3. COMPARISON THEOREMS BETWEEN TWO SIES

Several authors like Anderson [1], Ikeda and Watanabe [1] and Skorokhod [1] g comparison theorems for the solutions of two Itô's stochastic differential equations. common features of these theorems is that the diffusion coefficients of the equations are the same. O'Brien [1] gave a new comparison theorem for solutions of two Itô's equations with different diffusion terms which was then generalized by Gal'cuk and Davis [1]. However, both of O'Brien [1] and Gal'cuk and Davis [1] dealt with the homogeneous stochastic differential equation. Our purpose in this section is to give several comparison theorems for the solutions of more general SIES.

Let M be a continuous semimartingale with initial condition $M(0) = 0$. For the convenience, we write $\langle M^c, M^c\rangle(t) = \langle M^c\rangle_t$. Let Ψ denote the family of all progressive processes ψ such that

$$\int_0^t |\psi(s)|\, d\langle M^c\rangle_s < \infty \quad \text{a.s.} \quad \text{for all } t \geq 0$$

Denote by Φ the family of all progressive processes φ such that

$$\int_0^t (\varphi(s))^2\, d\langle M^c\rangle_s < \infty \quad \text{a.s.} \quad \text{for all } t \geq 0$$

We also denote by $C^{2,1}(R\times R_+, R)$ the family of all functions $V(x, t) : R\times R \to R$ with continuous partial derivatives V_x, V_{xx} and V_t, and, particularly, $V_x \neq 0$.

Theorem 4.3.1. Suppose there exist

(1) V^i $(i=1, 2) \in C^{2,1}(R\times R_+, R)$ with either $V^1_x > 0$ or $V^2_x > 0$ such that for given $x^1, x^2 \in R$

$$V^1(x, t) - V^1(x^1, 0) \geq V^2(x, t) - V^2(x^2, 0), \quad (x, t) \in R\times R_+ \tag{4.3.1}$$

(2) ψ^i $(i = 1, 2) \in \Psi$ such that

$$\int_0^t \psi^1(s)\, d\langle M^c\rangle_s \geq \int_0^t \psi^2(s)\, d\langle M^c\rangle_s \quad \text{a.s.} \quad t \geq 0 \tag{4.3.2}$$

(3) $\varphi \in \Phi$.

If we denote by X^i $(i=1, 2)$ the solution of the equation

$$X(t) = x^i + \int_0^t \frac{1}{V^i_x(X^i(s), \langle M^c\rangle_s)}\, \varphi(s)\, dM(s) -$$

$$- \int_0^t \frac{1}{V^i_x(X^i(s), \langle M^c\rangle_s)} \Bigg\{ \psi^i(s) + V^i_t(X^i(s), \langle M^c\rangle_s)$$

$$+ \frac{V^i_{xx}(X^i(s), \langle M^c\rangle_s)\,(\varphi(s))^2}{2\left[V^i_x(X^i(s), \langle M^c\rangle_s)\right]^2} \Bigg\} d\langle M^c\rangle_s \qquad (4.3.3)$$

then

$$X^1(t) \le X^2(t) \quad \text{a.s.} \qquad \text{for all } t \ge 0 \qquad (4.3.4)$$

Proof. Applying Itô's formula to $V^i(X^i(t), \langle M^c\rangle_t)$ we get

$$V^i(X^i(t), \langle M^c\rangle_t) - V^i(x^i, 0) = \int_0^t \varphi(s)\, dM(s) - \int_0^t \psi^i(s)\, d\langle M^c\rangle_s \qquad (4.3.5)$$

which, together with (4.3.2), implies

$$V^1(X^1(t), \langle M^c\rangle_t) - V^1(x^1, 0) \le V^2(X^2(t), \langle M^c\rangle_t) - V^2(x^2, 0) \quad \text{a.s.} \qquad (4.3.6)$$

for all $t \ge 0$. In the case $V^1_x > 0$, we get from assumption (4.3.1) that

$$V^1(X^2(t), \langle M^c\rangle_t) - V^1(x^1, 0) \ge V^2(X^2(t), \langle M^c\rangle_t) - V^2(x^2, 0) \quad \text{a.s.}$$

Consequently

$$V^1(X^1(t), \langle M^c\rangle_t) \le V^1(X^2(t), \langle M^c\rangle_t) \quad \text{a.s.}$$

which implies (4.3.4) immediately. Similarly, if $V^2_x > 0$, we get

$$V^1(X^1(t), \langle M^c\rangle_t) - V^1(x^1, 0) \ge V^2(X^1(t), \langle M^c\rangle_t) - V^2(x^2, 0) \quad \text{a.s.}$$

It follows

$$V^2(X^1(t), \langle M^c\rangle_t) \le V^2(X^2(t), \langle M^c\rangle_t) \quad \text{a.s.}$$

which also implies desired result (4.3.4). The proof is complete.

As a simple application of this theorem, we can get the following corollary due to Gal'cuk and Davis [1].

Corollary 4.3.2. Suppose

(1) σ^i $(i=1,2) : R \to (0,\infty)$ are continuously differentiable such that for given x^1, $x^2 \in R$

$$\int_{x^1}^{x} \frac{ds}{\sigma^1(s)} \geq \int_{x^2}^{x} \frac{ds}{\sigma^2(s)}, \qquad x \in R$$

(2) $\psi \in \Psi$.

If we denote by X^i $(i=1, 2)$ the solution of the equation

$$X^i(t) = x^i + \int_0^t \sigma^i(X^i(s))\, dM(s)$$

$$- \int_0^t \sigma^i(X^i(s)) \left\{ \psi(s) - \frac{1}{2} \sigma^i_x(X^i(s)) \right\} d\langle M^c\rangle_s$$

Then

$$X^1(t) \leq X^2(t) \text{ a.s.} \qquad \text{for all } t \geq 0$$

This corollary follows by setting $\varphi(t) = 1$, $\psi^1(t) = \psi^2(t) = \psi(t)$ and

$$V^i(x,t) = \int_{x^i}^{x} \frac{ds}{\sigma^i(s)}$$

in Theorem 4.3.1. Furthermore, if we let $\psi^1(t) = \psi^2(t) = 0$ and M be a Wiener process in Corollary 4.3.2, we get the comparison theorem of O'Brien [1].

Theorem 4.3.3. Suppose $V \in C^{2,1}(R \times R_+, R)$ with $V_x > 0$ and $\psi \in \Psi$, $\varphi \in \Phi$, $x_0 \in R$. Let $\xi(t)$ and $X(t)$ denote the solutions of the equations

$$\xi(t) = x_0 + \int_0^t \frac{1}{V_x(\xi(s), \langle M^c\rangle_s)}\, \varphi(s)\, dM(s)$$

$$- \int_0^t \frac{1}{V_x(\xi(s), \langle M^c\rangle_s)} \left\{ \psi(s) + V_t(\xi(s), \langle M^c\rangle_s) \right\} d\langle M^c\rangle_s), \quad t \geq 0 \qquad (4.3.7)$$

and

$$X(t) = x_0 + \int_0^t \frac{1}{V_x(X(s), <M^c>_s)} \varphi(s)\, dM(s)$$

$$- \int_0^t \frac{1}{V_x(X(s), <M^c>_s)} \Bigg\{ \psi(s) + V_t(\xi(s), <M^c>_s) +$$

$$+ \frac{V_{xx}(X(s), <M^c>_s)\,(\varphi(s))^2}{2\big[V_x(X(s), <M^c>_s)\big]^2} \Bigg\} d<M^c>_s), \quad t \geq 0 \qquad (4.3.8)$$

respectively. Then

$$\xi(t) \geq X(t) \quad \text{a.s.} \quad \text{for all } t \geq 0 \qquad (4.3.9)$$

if $V_{xx} \geq 0$, and

$$\xi(t) \leq X(t) \quad \text{a.s.} \quad \text{for all } t \geq 0 \qquad (4.3.10)$$

if $V_{xx} \leq 0$.

Proof. Using Itô's formula we deduce

$$V(\xi(t), <M^c>_t) - V(x_0, 0) = \int_0^t \varphi(s)\, dM(s) - \int_0^t \psi(s)\, d<M^c>_s$$

$$+ \int_0^t \frac{V_{xx}(\xi(s), <M^c>_s)\,(\varphi(s))^2}{2\big[V_x(\xi(s), <M^c>_s)\big]^2} d<M^c>_s \qquad (4.3.11)$$

If $V_{xx} \geq 0$, we get

$$V(\xi(t), <M^c>_t) - V(x_0, 0) \geq \int_0^t \varphi(s)\, dM(s) - \int_0^t \psi(s)\, d<M^c>_s$$

$$\geq V(X(t), <M^c>_t) - V(x_0, 0)$$

i.e.

$$V(\xi(t), <M^c>_t) \geq V(X(t), <M^c>_t)$$

which implies desired result (4.3.10). Similarly, if $V_{xx} \leq 0$, we can get (4.3.11).

The proof has completed.

The following corollary follows immediately by combining Theorems 4.3.1 and 4.3.3.

Corollary 4.3.4. With the assumptions of Theorem 4.3.1 we assume furthermore that

$$V_x^1 > 0, \quad V_x^2 > 0, \quad V_{xx}^1 \leq 0, \quad V_{xx}^2 \geq 0$$

Denote by X^i $(i=1, 2)$ the solution of the equation

$$X^i(t) = x^i + \int_0^t \frac{1}{V_x^i(X^i(s), <M^c>_s)} \varphi(s)\, dM(s) +$$

$$- \int_0^t \frac{1}{V_x^i(X^i(s), <M^c>_s)} \left\{ \psi^i(s) + V_t^i(X^i(s), <M^c>_s) \right\} d<M^c>_s$$

Then

$$X^1(t) \leq X^2(t) \quad \text{a.s.} \qquad \text{for all } t \geq 0$$

We now give some examples to illustrate our results.

Example 4.3.5. Let $X^i(t)$ $(i = 1, 2)$ denote the solutions of the equations

$$X^i(t) = x_0 + M(t) - \int_0^t (X^i(s) + \psi^i(s))\, d<M^c>_s$$

where $x_0 \in R$ and $\psi^i \in \Psi$ such that

$$\infty > \int_0^t \psi^1(s) \exp(<M^c>_s)\, d<M^c>_s \geq \int_0^t \psi^2(s) \exp(<M^c>_s)\, d<M^c>_s > -\infty \text{ a.s.}$$

for all $t \geq 0$. If we define

$$V^1(x, t) = V^2(x, t) = x e^t, \qquad \varphi(t) = \exp(-<M^c>_s)$$

for $x \in R$ and $t \geq 0$, it is easy to show the conditions of Theorem 4.3.1 hold so that we have

$$X^1(t) \leq X^2(t) \qquad \text{a.s.} \quad \text{for all } t \geq 0$$

Example 4.3.6. Let $X^1(t)$ and $X^2(t)$ be the solutions of the following equations

$$X^1(t) = x_o + \int_0^t \exp(-X^1(s))\, dM(s)$$

$$- \int_0^t \left\{ \frac{1}{1+\langle M^c\rangle_s} + \frac{1}{2}\exp(-2X^1(s)) \right\} d\langle M^c\rangle_s$$

and

$$X^2(t) = e^{x_o} + M(t) - \int_0^t \frac{1}{1+\langle M^c\rangle_s} X^2(s)\, d\langle M^c\rangle_s$$

respectively, where $x_o \in R$. Construct

$$V^1(x,t) = (1+t)\, e^x, \qquad V^2(x,t) = (1+t)\, x$$

$$\varphi(t) = 1+\langle M^c\rangle_t, \qquad \psi^1(t) = \psi^2(t) = 0$$

for $x \in R$ and $t \geq 0$. It is easy to check

$$V^1(x,t) - V^1(x_o, 0) \geq V^2(x,t) - V^2(e^{x_o}, 0)$$

Therefore, we have by Theorem 4.3.1 that

$$X^1(t) \leq X^2(t) \quad \text{a.s.} \quad \text{for all } t \geq 0$$

Example 4.3.7. Let $X^1(t)$ and $X^2(t)$ be the solutions of the following equations

$$X^1(t) = x^1 + \int_0^t \frac{1+\langle M^c\rangle_s}{1-b\sin(X^1(s))}\, dM(s)$$

$$+ \int_0^t \left\{ \frac{X^1(s) + b\cos(X^1(s))}{(1+\langle M^c\rangle_s)(1-b\sin(X^1(s)))} + \frac{(1+\langle M^c\rangle_s)^2\, b\cos(X^1(t))}{2(1-b\sin(X^1(s)))^3} \right\} d\langle M^c\rangle_s$$

and

$$X^2(t) = x^2 + \int_0^t (1 + \langle M^c \rangle_s)\, dM(s) + \int_0^t \frac{X^2(s)}{1 + \langle M^c \rangle_s}\, d\langle M^c \rangle_s$$

respectively, where $0 < b < 1$. We now construct

$$V^1(x, t) = \frac{x + b \cos x}{1 + t}, \qquad V^2(x, t) = \frac{x}{1 + t}$$

$$\varphi(t) = 1, \qquad \psi^1(t) = \psi^2(t) = 0$$

for $x \in R$ and $t \geq 0$. If $x^1 = x^2 = \pi$, we get

$$V^1(x, t) - V^1(x^1, 0) - V^2(x, t) + V^2(x^2, 0)$$

$$= \frac{1}{1+t} b \cos x - b \cos \pi \geq 0$$

Hence, by Theorem 4.3.1, we have

$$X^1(t) \leq X^2(t) \quad \text{a.s.} \quad \text{for all } t \geq 0$$

If $x^1 = x^2 = 0$, conversely, we get

$$V^1(x, t) - V^1(x^1, 0) - V^2(x, t) + V^2(x^2, 0)$$

$$= \frac{1}{1+t} b \cos x - b \cos 0 \leq 0$$

So it follows from Theorem 4.3.1 that

$$X^1(t) \geq X^2(t) \quad \text{a.s.} \quad \text{for all } t \geq 0$$

Example 4.3.8. Consider the following two equations

$$\xi(t) = x_0 + \int_0^t \frac{1}{1 + \langle M^c \rangle_s} \exp\left\{ - (1 + \langle M^c \rangle_s)\, \xi(s) \right\} dM(s)$$

$$+ \int_0^t \frac{1}{1 + \langle M^c \rangle_s}\, \xi(s)\, d\langle M^c \rangle_s$$

and

$$X(t) = x_0 + \int_0^t \frac{1}{1+\langle M^c\rangle_s} \exp\Big\{-(1+\langle M^c\rangle_s)\, X(s)\Big\} \; M(s)$$

$$+ \int_0^t \frac{1}{1+\langle M^c\rangle_s} \Big\{X(s) + \frac{1}{2}\exp\Big[-2(1+\langle M^c\rangle_s)\, X(s)\Big]\; \langle M^c\rangle_s$$

where $x_0 \in R$. If we let

$$V(x,t) = e^{(1+t)\,x}, \qquad \varphi(t) = 1, \qquad \psi(t) = 0$$

for $x \in R$ and $t \geq 0$, then $V_x > 0$ and $V_{xx} > 0$. Therefore we have

$$\xi(t) \geq X(t) \quad \text{a.s.} \quad \text{for all } t \geq 0.$$

by Theorem 4.3.3.

Example 4.3.9. We finally consider the following two equations

$$\xi(t) = x_0 + \int_0^t e^{\xi(s)}\, dM(s) - \int_0^t \frac{\cos\langle M^c\rangle_s}{2 + \sin\langle M^c\rangle_s} d\langle M^c\rangle$$

and

$$X(t) = x_0 + \int_0^t e^{X(s)}\, dM(s) + \int_0^t \left\{ \frac{1}{2} e^{2X(s)} - \frac{\cos\langle M^c\rangle_s}{2 + \sin\langle M^c\rangle_s} \right\} d\langle M^c\rangle_s$$

Let

$$V(x,t) = -(2+\sin t)\, e^{-x}, \qquad \varphi(t) = 2 + \sin\langle M^c\rangle_t, \qquad \psi(t) = 0$$

for $x \in R$ and $t \geq 0$. We then have $V_x > 0$ and $V_{xx} < 0$. Consequently, applying Theorem 4.3.3 we get

$$\xi(t) \leq X(t) \quad \text{a.s.} \quad \text{for all } t \geq 0.$$

Chapter V

Transformation Formula of Stochastic Integrals

5.1. INTRODUCTION

It is well known that stochastic integrals were first introduced by Ito§ [1] in 1942 to formulate the stochastic differential equation that determined Kolmogorov's diffusion processes (cf. Kolmogorov [1]). It was Doob [1] who pointed out the martingale character of stochastic integrals and suggested that a unified theory on stochastic integrals should be established in the framework of martingale theory. His program was accomplished by Fisk [1], Corrège [1], Kunita and Watanabe [1] and Meyer [2]. Obviously, the stochastic integral with respect to martingales is an extension of classical Itô's integral with respect to Brownian motions. A natural question arises: is it possible to transform stochastic integrals with respect to semimartingales into Itô's integrals?

We shall establish the transformation formula. It should be pointed out that the transformation formula is nothing but the well-known technique of random time-change. The roots of this lie in the famous paper of Dubins and Schwartz [1], on time-changes of martingales, and for example the technique as applied to Brownian integrals is to be found in the celebrated monograph of McKean [1] and recent book of Karatzas and Shreve [1]. We will apply this transformation formula to study the properties of SIES, for instance, the stochastic stability and boundedness. Particularly, we discuss the Markov property of the solutions to SIES. This formula is also employed to study the exponential stability of the solutions to the linear delay SIES. Some results in this chapter were presented at the 17th Conference on Stochastic Processes and Their Applications, 27 June - 1 July 1988, Rome, Italy. The results in Section 5.4 appeared in the Proceedings of the 1989 IEEE International Conference on Control and Applications, 4-6 April 1989, Israel.

5.2. TRANSFORMATION FORMULA

Let $\mathcal{M}_{c,loc}$ denote the family of all continuous local martingales. If $M \in \mathcal{M}_{c,loc}$, denote by $\mathcal{L}^2_{loc}(M)$ the set of all predictable processes X such that

$$\int_0^t (X(s))^2 \, d\langle M\rangle_s < \infty \quad \text{a.s.} \quad \text{for all } t \geq 0$$

where $\langle M\rangle_s = \langle M, M\rangle(s)$. Let $\mathcal{Q}$ stand for the family of all adapted left continuous step processes with deterministic (independent of $\omega \in \Omega$) jump points.

We can easily check that $\mathcal{Q} \subset \mathcal{L}^2_{loc}(M)$ for any $M \in \mathcal{M}_{c,loc}$ and that for every $X \in \mathcal{L}^2_{loc}(M)$ there exists a sequence of processes $\{X^n\} \in \mathcal{Q}$ such that

$$\lim_{n\to\infty} \int_0^t (X^n(s) - X(s))^2 \, d\langle M\rangle_s = 0 \quad \text{a.s.} \quad \text{for all } t \geq 0$$

Let $\mathbb{T}$ denote the set of all finite $\{\mathcal{F}_t\}$-stopping times.

Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$. Define, for all $t \geq 0$,

$$T_t = \inf\{ s \geq 0 : \langle M\rangle_s > t \}, \qquad T_{t-} = \inf\{ s \geq 0 : \langle M\rangle_s \geq t \}$$

$$\mathcal{G}_t = \mathcal{F}_{T_t} \qquad \text{and} \qquad N(t) = M(T_t)$$

We then have the following propositions (cf. Yan [1], pp.26, 92, 254):

(1) $T_{t-} = \lim_{s\uparrow t} T_{s-} = \lim_{s\uparrow t} T_s = \sup\{ s \geq 0 : \langle M\rangle_s < t \}$ for all $t > 0$.

(2) $T_t = \lim_{s\downarrow t} T_s = \lim_{s\downarrow t} T_{s-}$ for all $t \geq 0$.

(3) $\langle M\rangle_t = \inf\{ s : T_s > t \} = \inf\{ s : T_{s-} > t \}$

$= \sup\{ s : T_s \leq t \} = \sup\{ s : T_{s-} \leq t \}$ for all $t \geq 0$

(4) $\langle M\rangle_{T_t} = \langle M\rangle_{T_{t-}} = t$ a.s. on $\{ \omega : t < \langle M\rangle_\infty \}$ for all $t \geq 0$, where $\langle M\rangle_\infty := \lim_{s\to\infty} \langle M\rangle_s$.

(5) T_t is a $\{\mathcal{F}_t\}$-stopping time and T_{t-} is a $\{\mathcal{F}_t\}$-predictable time for all $t \geq 0$.

(6) $T_{\langle M\rangle_t -} \leq t \leq T_{\langle M\rangle_t}$ a.s. for all $t \geq 0$.

(7) If $S \in \mathbb{T}$, $\langle M\rangle_S$ is a $\{\mathcal{G}_t\}$-stopping time.

(8) If S is a $\{\mathcal{G}_t\}$-stopping time, T_S is a $\{\mathcal{F}_t\}$-stopping time and $\mathcal{G}_S = \mathcal{F}_{T_S}$.

(9) $N := (N(t))$ is a continuous local martingale relative to $\{\mathcal{G}_t\}$ and $\langle N\rangle_t = t$ a.s. on $\{\omega : t < \langle M\rangle_\infty\}$ for all $t \geq 0$. Particularly, $(N(t \wedge \langle M\rangle_S))_{t\geq 0}$ is a stopping process of a Wiener process relative to $\{\mathcal{G}_t\}$ for any $S \in \mathbb{T}$.

(10) There exists a sequence $\{S_n\}$ of strictly positive $\{\mathcal{G}_t\}$-stopping times such that $\{(\omega, t) : T_{t-} \neq T_t\} \subset \bigcup_n [\![S_n]\!]$ (cf. Yan [1], Theorem 5.20).

These propositions will be used without any explanation afterwards. In order to prove our transformation formula for stochastic integrals with respect to martingales, we need to prove the following lemma first.

Lemma 5.2.1. Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$ and $X \in \mathcal{Q}$. Then for any $\tau \in \mathbb{T}$ we have

$$\int_0^\tau X(t)\, dM(t) = \int_0^{\langle M\rangle_\tau} X(T_{t-})\, dN(t) \tag{5.2.1}$$

where the left hand side integral and right hand side integral are relative to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t\}$ respectively.

Proof. Obviously, it is enough to prove (5.2.1) for $X = \xi \mathbb{1}_{(a,b]}$, where $0 \leq a < b < \infty$ and ξ is $\mathcal{F}_a$-measurable random variable. In this case, we deduce

$$\int_0^{\langle M\rangle_\tau} \xi \mathbb{1}_{(a,b]}(T_{t-})\, dN(t)$$

$$= \int_0^\infty \xi \mathbb{1}_{]\!] \inf\{s:\, T_{s-} > a\},\ \sup\{s:\, T_{s-} \leq b\}]\!]}(t)\, \mathbb{1}_{]\!] 0, \langle M\rangle_\tau]\!]}(t)\, dN(t)$$

$$= \int_0^\infty \xi \mathbb{1}_{]\!] \langle M\rangle_a, \langle M\rangle_b]\!]}(t)\, \mathbb{1}_{]\!] 0, \langle M\rangle_\tau]\!]}(t)\, dN(t)$$

$$= \int_0^\infty \xi\, \mathbb{1}_{]\!]\langle M\rangle_{\tau\wedge a},\, \langle M\rangle_{\tau\wedge b}]\!]}(t)\, \mathbb{1}_{]\!]0,\langle M\rangle_\tau]\!]}(t)\, dN(t)$$

$$= \int_0^\infty \xi\, \mathbb{1}_{\{a\le\tau\}}\, \mathbb{1}_{]\!]\langle M\rangle_{\tau\wedge a},\, \langle M\rangle_{\tau\wedge b}]\!]}(t)\, \mathbb{1}_{]\!]0,\langle M\rangle_\tau]\!]}(t)\, dN(t) \tag{5.2.2}$$

Since $\xi\, \mathbb{1}_{\{a\le\tau\}}$ is both $\mathcal{F}_\tau$-measurable and $\mathcal{F}_a$-measurable, $\xi\, \mathbb{1}_{\{a\le\tau\}}$ is $\mathcal{F}_{\tau\wedge a}$-measurable. Noticing

$$\mathcal{G}_{\langle M\rangle_{\tau\wedge a}} = \mathcal{F}_{T_{\langle M\rangle_{\tau\wedge a}}} \supset \mathcal{F}_{\tau\wedge a}$$

we imply $\xi\, \mathbb{1}_{\{a\le\tau\}}$ is $\mathcal{G}_{\langle M\rangle_{\tau\wedge a}}$-measurable. Hence, it follows from (5.2.2) that

$$\int_0^{\langle M\rangle_\tau} \xi\, \mathbb{1}_{(a,b]}(T_{t-})\, dN(t) = \xi\, \mathbb{1}_{\{a\le\tau\}} \left\{ N(\langle M\rangle_{\tau\wedge b}) - N(\langle M\rangle_{\tau\wedge a}) \right\} \tag{5.2.3}$$

However $N(\langle M\rangle_{\tau\wedge b}) = M(T_{\langle M\rangle_{\tau\wedge b}})$ and $\langle M\rangle_{T_{\langle M\rangle_{\tau\wedge b}}} = \langle M\rangle_{\tau\wedge b}$. These, together with Theorem 8.40 of Yan [1], yield

$$N(\langle M\rangle_{\tau\wedge b}) = M(\tau\wedge b) \tag{5.2.4}$$

Similarly

$$N(\langle M\rangle_{\tau\wedge a}) = M(\tau\wedge a) \tag{5.2.5}$$

Therefore, we have from (5.2.3) that

$$\int_0^{\langle M\rangle_\tau} \xi\, \mathbb{1}_{(a,b]}(T_{t-})\, dN(t) = \xi\, \mathbb{1}_{\{a\le\tau\}} \left\{ M(\tau\wedge b) - M(\tau\wedge a) \right\}$$

$$= \int_0^\infty \xi\, \mathbb{1}_{\{a\le\tau\}}\, \mathbb{1}_{]\!]\tau\wedge a,\, \tau\wedge b]\!]}(t)\, dM(t)$$

$$= \int_0^\tau \xi\, \mathbb{1}_{(a,\, b]}(t)\, dM(t)$$

The lemma has been proved.

We now can prove our transformation formula.

Theorem 5.2.2. Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$ and $X \in \mathcal{L}^2_{loc}(M)$. Then for any $\tau \in \mathbb{T}$ we have

$$\int_0^\tau X(t)\, dM(t) = \int_0^{\langle M \rangle_\tau} X(T_{t-})\, dN(t) = \int_0^{\langle M \rangle_\tau} X(T_t)\, dN(t) \tag{5.2.6}$$

where the first integral and the other two integrals are relative to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t\}$ respectively.

Proof. Let $\{ X^n \} \subset \mathcal{Q}$ such that

$$\lim_{n\to\infty} \int_0^t (X^n(t) - X(t))^2\, d\langle M \rangle_t = 0 \quad \text{a.s.} \quad \text{for all } t \geq 0 \tag{5.2.7}$$

Hence, for any $\tau \in \mathbb{T}$,

$$\int_0^\tau (X^n(t) - X(t))^2\, d\langle M \rangle_t \xrightarrow{P} 0 \qquad \text{as } n \to \infty \tag{5.2.8}$$

This, together with Theorem 9.35 of Yan [8], yields

$$\int_0^\tau X^n(t)\, dM(t) \xrightarrow{P} \int_0^\tau X(t)\, dM(t) \qquad \text{as } n \to \infty \tag{5.2.9}$$

On the other hand, we have

$$\int_0^{\langle M \rangle_\tau} (X^n(T_{t-}) - X(T_{t-}))^2\, d\langle N \rangle_t = \int_0^{\langle M \rangle_\tau} (X^n(T_{t-}) - X(T_{t-}))^2\, dt$$

$$= \int_0^\tau (X^n(t) - X(t))^2\, d\langle M \rangle_t \xrightarrow{P} 0 \qquad \text{as } n \to \infty$$

where Lemma 3.2.1 has been used. Thus

$$\int_0^{\langle M\rangle_\tau} X^n(T_{t-})\, dN(t) \xrightarrow{P} \int_0^{\langle M\rangle_\tau} X(T_{t-})\, dN(t) \qquad \text{as } n \to \infty \tag{5.2.10}$$

In view of Lemma 5.2.1, we also have

$$\int_0^{\tau} X^n(t)\, dM(t) = \int_0^{\langle M\rangle_\tau} X^n(T_{t-})\, dN(t)$$

which, together with (5.2.9) and (5.2.10), deduces

$$\int_0^{\tau} X(t)\, dM(t) = \int_0^{\langle M\rangle_\tau} X(T_{t-})\, dN(t) \tag{5.2.11}$$

Finally, noticing $(N(t))$ is continuous and that there exists a sequence of $\{S_n\}$ of strictly positive $\{\mathfrak{G}_t\}$-stopping times such that $\{ (\omega, t) : T_{t-} \neq T_t \} \subset \bigcup_n [\![S_n]\!]$, we deduce desired result (5.2.6) from (5.2.11) immediately. The proof is complete.

Remark 5.2.3. We know $(N(t \wedge \langle M\rangle_\tau))_{t\geq 0}$ is a stopping process of a Wiener process relative to $\{\mathfrak{G}_t\}$ for any $\tau \in \mathbb{T}$ if $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$. Therefore, Theorem 5.2.2 means the stochastic integrals with respect to M can be transformed into classical Itô's integrals. Since the hypothesis $M_0 = 0$ is not essential (in fact, we can consider $M - M_0$ if necessary), we have formulated that the stochastic integrals with respect to continuous local martingales can be transformed into classical Itô's integrals.

5.3. APPLICATIONS TO SIES

In this section we will use the transformation formula to discuss the properties of solutions to stochastic integral equations with respect to continuous local martingales, for example, the Markovian property and the asymptotic property.

Throughout this section, we let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$ a.s. Let us consider a stochastic integral equation

$$X(t) = x_0 + \int_0^t f(.,s,X(s))\,d\langle M\rangle_s + \int_0^t g(.,s,X(s))\,dM_s, \quad t \geq 0 \qquad (5.3.1)$$

where $x_0 \in R^n$ and $f, g : \Omega \times R_+ \times R^n \to R^n$ satisfy the conditions for the existence and uniqueness of the solutions discussed in Section 1.5. Thus, there exists a unique solution to Eq.(5.3.1) which is a continuous $\{\mathcal{F}_t\}$-adapted process. In view of Theorem 5.2.2 and Lemma 3.2.1, we get from (5.3.1) that

$$X(T_t) = x_0 + \int_0^{T_t} f(.,s,X(s))\,d\langle M\rangle_s + \int_0^{T_t} g(.,s,X(s))\,dM_s$$

$$= x_0 + \int_0^t f(.,T_s,X(T_s))\,ds + \int_0^t g(.,T_s,X(T_s))\,dN_s \qquad (5.3.2)$$

for all $t \geq 0$. If we set

$$\bar{f}(.,t,x) = f(.,T_t,x) \quad \text{and} \quad \bar{g}(.,t,x) = g(.,T_t,x)$$

then (5.3.2) means that the continuous $\{\mathcal{G}_t\}$-adapted process $(X(T_t))_{t\geq 0}$ is the unique solution of Itô's equation

$$Y(t) = x_0 + \int_0^t \bar{f}(.,s,Y(s))\,ds + \int_0^t \bar{g}(.,s,Y(s))\,dN_s, \quad t \geq 0 \qquad (5.3.3)$$

Hence we have already proved

Theorem 5.3.1. Let $(X(t))_{t\geq 0}$ be the unique solution of Eq.(5.3.1). Then $(X(T_t))_{t\geq 0}$ is the unique solution of Eq.(5.3.3). Particularly, $(X(T_t))_{t\geq 0}$ is a continuous strong Markov process relative to $\{\mathcal{G}_t\}$.

We now prove another useful theorem.

Theorem 5.3.2. The solution of Eq.(5.3.1) satisfies

$$X(t) = X(T_{\langle M\rangle_t}) \quad \text{a.s.} \qquad t \geq 0 \qquad (5.3.4)$$

Proof. Noticing $T_{\langle M\rangle_t}$ is a $\{\mathcal{F}_t\}$-stopping time and $t \leq T_{\langle M\rangle_t}$ a.s. for all $t \geq 0$, we deduce from (5.3.1) that

$$X(T_{\langle M\rangle_t}) - X(t) = \int_t^{T_{\langle M\rangle_t}} f(., s, X(s))\, d\langle M\rangle_s + \int_t^{T_{\langle M\rangle_t}} g(., s, X(s))\, dM_s$$

However, $\langle M\rangle_t = \langle M\rangle_s$ if $t \leq s \leq T_{\langle M\rangle_t}$. Therefore

$$\int_t^{T_{\langle M\rangle_t}} f(., s, X(s))\, d\langle M\rangle_s = 0 \quad \text{a.s.}$$

and

$$E\Big(\int_t^{T_{\langle M\rangle_t}} g(., s, X(s))\, dM_s\Big)^2 = E\int_t^{T_{\langle M\rangle_t}} \big(g(., s, X(s))\big)^2\, d\langle M\rangle_s = 0$$

from which it follows that

$$\int_t^{T_{\langle M\rangle_t}} g(., s, X(s))\, dM_s = 0 \quad \text{a.s.}$$

Combining these we arrive at

$$X(T_{\langle M\rangle_t}) = X(t) \quad \text{a.s.}$$

which is the desired result.

By using these results, together with the well-known properties of the solutions of Itô's equations, it is very convenient to discuss the properties of the solutions to Eq.(5.3.1). We give some examples to illustrate our idea.

Example 5.3.3. Consider the one-dimensional linear stochastic system

$$X(t) = x_0 + \int_0^t a\, X(s)\, d\langle M\rangle_s + \int_0^t b\, X(s)\, dM_s, \quad t \geq 0 \tag{5.3.5}$$

where $x_0 \neq 0$ and a, b are constants with $b \neq 0$. We know (cf. Arnold [1]) that for the solution of Itô's equation

$$Y(t) = x_0 + \int_0^t a\,Y(s)\,ds + \int_0^t b\,Y(s)\,dN_s, \quad t \geq 0 \tag{5.3.6}$$

we have

$$\lim_{t\to\infty} Y(t) = 0 \quad \text{a.s.} \quad \text{if } a < b^2/2$$

$$\overline{\lim_{t\to\infty}}\, Y(t) = \infty \quad \text{a.s.} \quad \text{if } a \geq b^2/2$$

$$\underline{\lim_{t\to\infty}}\, Y(t) = -\infty \quad \text{a.s.} \quad \text{if } a = b^2/2$$

Hence, an application of Theorems 5.3.1 and 5.3.2 yields that for the solution of Eq.(5.3.5) we have

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} X(T_t) = 0 \quad \text{a.s.} \quad \text{if } a < b^2/2$$

$$\overline{\lim_{t\to\infty}}\, X(t) = \overline{\lim_{t\to\infty}}\, X(T_t) = \infty \quad \text{a.s.} \quad \text{if } a \geq b^2/2$$

$$\underline{\lim_{t\to\infty}}\, X(t) = \underline{\lim_{t\to\infty}}\, X(T_t) = -\infty \quad \text{a.s.} \quad \text{if } a = b^2/2$$

Example 5.3.4. Assume a(.) and b(.) are bounded real functions defined on R_+. Consider the one-dimensional stochastic integral equation

$$X(t) = x_0 + \int_0^t a(\langle M\rangle_s)\,X(s)\,d\langle M\rangle_s + \int_0^t b(\langle M\rangle_s)\,X(s)\,dM_s, \quad t \geq 0 \tag{5.3.7}$$

Let $p > 0$. We know (cf. Arnold [1]) that for the solution of Itô's equation

$$Y(t) = x_0 + \int_0^t a(s)\,Y(s)\,ds + \int_0^t b(s)\,Y(s)\,dN_s, \quad t \geq 0 \tag{5.3.8}$$

we have

$$\lim_{t\to\infty} E\,|\,Y(t)\,|^p = 0 \tag{5.3.9}$$

if and only if

$$\lim_{t\to\infty} \int_0^t (p\, a(s) + p(p-1)\, b^2(s)/2)\, ds = -\infty \tag{5.3.10}$$

Therefore, $\lim\limits_{t\to\infty} E\,|\,X(T_t)\,|^p = 0$ if and only if (5.3.10) holds.

Example 5.3.5. We now consider the two-dimensional equation

$$X(t) = x_o + \int_0^t f(s, X(s))\, d\langle M\rangle_s + \int_0^t g(s, X(s))\, dM_s, \quad t \geq 0 \tag{5.3.11}$$

where

$$f(t, x) = \begin{bmatrix} -(x_1 + x_2)(1 + e^{-t}) \\ -(x_2 - x_1)(1 + e^{-t}) \end{bmatrix}, \quad g(t, x) = \begin{bmatrix} g_1(t, x) \\ g_2(t, x) \end{bmatrix}$$

Assume $g(t, 0) = 0$ and

$$(g_1(t, x) + g_2(t, x))^2 \leq (x_1 + x_2)^2$$

$$(g_1(t, x) - g_2(t, x))^2 \leq (x_1 - x_2)^2$$

$$|\, g_i(t, x) - g_i(t, y)\,| \leq c\,|\, x - y\,|, \quad i = 1, 2$$

for all $(t, x) \in R_+ \times R^2$, where c is a positive constant. In order to discuss the stability of the solution we consider the corresponding Itô's equation

$$Y(t) = x_o + \int_0^t f(T_s, Y(s))\, ds + \int_0^t g(T_s, Y(s))\, dN_s, \quad t \geq 0 \tag{5.3.12}$$

We introduce Lyapunov's function

$$V(x) = |\,x\,|^2$$

Then, for Eq.(5.3.12) we have

$$V_x(x)\, f(T_t, x) + \frac{1}{2} g(T_t, x)'\, V_{xx}(x)\, g(T_t, x)$$

$$= \; - 2\{ x_1(x_1 + x_2) + x_2(x_2 - x_1) \} (1 - e^{-T_t}) \; + \; | g(T_t, x) |^2$$

$$\leq \; - 2 (1 - e^{-T_t}) | x |^2 \; + \; \frac{1}{2}((x_1 + x_2)^2 + (x_1 - x_2)^2)$$

$$\leq \; - | x |^2, \qquad (t, x) \in R_+ \times R^2$$

Therefore, we deduce (cf. Arnold [1] or Friedman [1]) that the trivial solution of Eq.(5.3.12) is stochastically asymptotically stable. In view of Theorems 5.3.1 and 5.3.2, we obtain that the trivial solution of Eq.(5.3.11) is also stochastically asymptoticallystable.

Example 5.3.6. We finally consider the stochastic boundedness of the solution to the two-dimensional stochastic equation

$$X(t) \; = \; x_0 + \int_0^t f(s, X(s)) \, d\langle M\rangle_s + \int_0^t g(s, X(s)) \, dM_s, \quad t \geq 0 \tag{5.3.13}$$

where

$$f(t, x) = \begin{bmatrix} -(x_1 + x_2) - e^{-\langle M\rangle_t} \\ \\ -(x_2 - x_1) - e^{-\langle M\rangle_t} \end{bmatrix}, \qquad g(t, x) = \begin{bmatrix} x_1 + e^{-\langle M\rangle_t} \\ \\ x_2 + e^{-\langle M\rangle_t} \end{bmatrix}$$

For the corresponding Itô's equation

$$Y(t) \; = \; x_0 + \int_0^t f(T_s, Y(s)) \, ds + \int_0^t g(T_s, Y(s)) \, dN_s, \quad t \geq 0 \tag{5.3.14}$$

we introduce Lyapunov's function

$$V(t, x) \; = \; | x |^2 + e^{-2t}$$

Thus

$$V_t(t, x) \; + \; V_x(x) \, f(T_t, x) \; + \; \frac{1}{2} g(T_t, x)' \, V_{xx}(x) \, g(T_t, x) \; = \; - | x |^2$$

Hence the solutions of Eq.(5.3.14) are stochastically ultimately bounded (cf. Li and Mao [3]), so are the solutions of Eq.(5.3.13).

5.4. STABILITY OF LINEAR DELAY SIES

This section is devoted to the study of the exponential stability in mean square of the solutions of the linear delay SIES. As we know, the general linear delay SIES have the form

$$X(t) = \varphi(t_0) + \int_{t_0}^{t} A\, X(s)\, d\mu(s) + \int_{t_0}^{t} B(s)\,(\,X(s) - X(s-\tau)\,)\, d\mu(s)$$

$$+ \sum_{i=1}^{m} \int_{t_0}^{t} C_i(s)\, X(s)\, dM_i(s) + \sum_{i=1}^{m} \int_{t_0}^{t} D_i(s)\,(X(s) - X(s-\tau)\,)\, dM_i(s), \quad t \ge t_0 \quad (5.4.1)$$

$$X(t) = \varphi(t), \qquad t_0 - \tau \le t \le t_0$$

If $C_i = 0$ $(1 \le i \le m)$ and $A(t) = A$, Eq.(5.4.1) reduces to

$$X(t) = \varphi(t_0) + \int_{t_0}^{t} A\, X(s)\, d\mu(s) + \int_{t_0}^{t} B(s)\,(\,X(s) - X(s-\tau)\,)\, d\mu(s)$$

$$+ \sum_{i=1}^{m} \int_{t_0}^{t} D_i(s)\,(X(s) - X(s-\tau)\,)\, dM_i(s), \quad t \ge t_0 \qquad (5.4.2)$$

$$X(t) = \varphi(t), \qquad t_0 - \tau \le t \le t_0$$

which might be regarded as a stochastic perturbed system of the following equation

$$X(t) = \varphi(t_0) + \int_{t_0}^{t} A\, X(s)\, d\mu(s), \quad t \ge t_0 \qquad (5.4.3)$$

An interesting and useful problem is: does the stability of Eq.(5.4.3) share by Eq.(5.4.2) if τ small enough? Can we find an estimate for such τ?

Although we have not yet got the answer for the general case since this problem, as we know, is extremely difficult, we have a perfect answer for the following slightly special linear delay SIES

$$X(t) = \varphi(t_o) + \int_{t_o}^{t} A\,X(s)\,d\mu(s) + \int_{t_o}^{t} B(s)\,(\,X(s) - e^{A\tau}X(s-\tau)\,)\,d\mu(s)$$

$$+ \sum_{i=1}^{m} \int_{t_o}^{t} D_i(s)\,(\,X(s) - e^{A\tau}X(s-\tau)\,)\,dM_i(s), \quad t \geq t_o \tag{5.4.4}$$

$$X(t) = \varphi(t), \quad t_o - \tau \leq t \leq t_o$$

(Please notice $e^{A\tau} \to I$ = the identity matrix as $\tau \to 0$). Indeed, in this section we prove that if the solutions of Eq.(5.4.3) are exponentially stable, then the solutions of Eq.(5.4.4) are exponentially stable in mean square provided τ is small enough. We also give an estimate for τ.

Throughout this section we let $A \in R^{n\times n}$, $B, D_i(1\leq i\leq m) : \Omega\times[\,0, \infty\,) \to R^{n\times n}$ be $\{\mathcal{F}_t\}$-progressive, τ a positive constant and $\varphi(t)$ an n-dimensional $\{\mathcal{F}_t\}$-progressive process, here we let $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [\,-\tau, 0\,]$. Let M_i ($i = 1, 2, \ldots, m$) be continuous local martingales with $M_i(0) = 0$ a.s. and μ be a continuous adapted increasing process with $\mu(0) = 0$. We assume

$$\langle M_i\rangle_t = \mu(t), \quad t \geq 0, \quad i = 1, 2, \ldots m \tag{5.4.5}$$

We first prove the following theorem.

Theorem 5.4.1. Assume

(1) there exists a positive number K such that

$$|B(t)|^2 \leq K \text{ a.s. and } |D_i(t)|^2 \leq K \text{ a.s.} \quad i = 1, \ldots, m \tag{5.4.6}$$

for all $t \geq 0$;

(2) there exist two positive constants λ and η such that

$$|e^{At}|^2 \leq \eta\, e^{-\lambda t}, \quad t \geq 0 \tag{5.4.7}$$

(which is equivalent to assume all of the characteristic roots of A have negative real parts);

(3) τ is so small that

$$\theta\tau e^{\theta\tau} \leq 1 \tag{5.4.8}$$

where

$$\theta := K(\,m+1\,)(\,\tau+m\,) \tag{5.4.9}$$

(4)
$$E \int_{t_o-\tau}^{t_o} |\varphi(s)|^2 \, d\mu(s+\tau) < \infty \tag{5.4.10}$$

(5) there exist $\alpha > 0$, $\delta \geq 0$ and $\rho(t) : R_+ \to R_+$ such that

$$\alpha t - \delta \leq \mu(t+s) - \mu(s) \leq \rho(t) \quad \text{a.s.} \quad \text{for all } t \geq 0 \text{ and } s \geq 0 \tag{5.4.11}$$

and

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log(\rho(t)) := \zeta < \infty \tag{5.4.12}$$

Then, for the solution of Eq.(5.4.4), we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|X(t)|^2 \leq -\alpha\lambda + 2\zeta \tag{5.4.13}$$

Remark 5.4.2. (1) In a lot of cases, ζ could be very small. For instant, $\zeta = 0$ if $\rho(t) = \beta t^p$, where $\beta > 0$ and $p \geq 1$.

(2) In applications, we can choose τ such that

$$\theta\tau = K(m+1)(\tau + m)\tau \leq 0.567143 \leq q_o$$

where q_o is the unique solution of the equation $q e^q = 1$.

To prove this theorem we first prove it holds in the case $M_i (1 \leq i \leq m)$ are Wiener processes, and then we use the transformation formula to prove it also holds for the general case. Hence, let us consider the following delay linear Itô 's equation

$$X(t) = \varphi(t_o) + \int_{t_o}^{t} A X(s)\,ds + \int_{t_o}^{t} B(s)\,(X(s) - e^{A\tau} X(s-\tau))\,ds$$

$$+ \sum_{i=1}^{m} \int_{t_o}^{t} D_i(s)\,(X(s) - e^{A\tau} X(s-\tau))\,dW_i(s), \quad t \geq t_o \tag{5.4.14}$$

$$X(t) = \varphi(t), \quad t_o - \tau \leq t \leq t_o$$

where $(W_1(t), \ldots, W_m(t))$ is an m-dimensional Wiener Process. For this equation we have the following lemma.

Lemma 5.4.3. Let conditions (1) - (3) in Theorem 5.4.1 hold. Assume further

$$\beta := \int_{t_o-\tau}^{t_o} E\,|\varphi(s)|^2\, ds < \infty \tag{5.4.15}$$

Then, for the solution of Eq.(5.4.14),

$$e^{\lambda t} E\,|\,X(t) - e^{A\tau} X(t-\tau)\,|^2 \le c_2, \quad t \ge t_o + \tau \tag{5.4.16}$$

where c_2 is a constant. Particularly, we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E\,|\,X(t)\,|^2 \le -\lambda \tag{5.4.17}$$

Proof. Step 1

For $t_o \le t \le t_o + \tau$, we have

$$X(t) = \varphi(t_o) + \int_{t_o}^{t} (A + B(s))\, X(s)\, ds - \int_{t_o}^{t} B(s)\, e^{A\tau} \varphi(s-\tau)\, ds$$

$$+ \sum_{i=1}^{m} \int_{t_o}^{t} D_i(s)\, X(s)\, dW_i(s) - \sum_{i=1}^{m} \int_{t_o}^{t} D_i(s)\, e^{A\tau} \varphi(s-\tau)\, dW_i(s)$$

which implies

$$E\,|\,X(t)\,|^2 \le (2m+3)\left\{ E|\,\varphi(t_o)\,|^2 + \eta\,\beta\, K(\tau + m) \right\}$$

$$+ (2m+3)\left[2\,\tau\,(\,|\,A\,|^2 + K) + mK \right] \int_{t_o}^{t} E\,|\,X(s)\,|^2\, ds$$

Using the Gronwall inequality (cf. [2]), we deduce

$$E\,|\,X(t)\,|^2 \le (2m+3)\left\{ E|\,\varphi(t_o)\,|^2 + \eta\,\beta\, K(\tau + m) \right\}$$

$$\cdot \exp\left\{ (2m+3)\left[2\,\tau\,(\,|\,A\,|^2 + K) + mK \right] \right\}$$

$$:= c_1, \qquad t_o \le t \le t_o + \tau \tag{5.4.18}$$

Step 2

By Corollary 2.4.2, we have, for $t \ge t_0$, that

$$e^{-At} X(t) = e^{-At_0} \varphi(t_0) + \int_{t_0}^{t} e^{-As} B(s) \left(X(s) - e^{A\tau} X(s-\tau) \right) ds$$

$$+ \sum_{i=1}^{m} \int_{t_0}^{t} e^{-As} D_i(s) \left(X(s) - e^{A\tau} X(s-\tau) \right) dW_i(s) \qquad (5.4.19)$$

Hence, for $t \ge t_0 + \tau$, we have

$$X(t) - e^{A\tau} X(t-\tau) = \int_{t-\tau}^{t} e^{A(t-s)} B(s) \left(X(s) - e^{A\tau} X(s-\tau) \right) ds$$

$$+ \sum_{i=1}^{m} \int_{t-\tau}^{t} e^{A(t-s)} D_i(s) \; X(s) - e^{A\tau} X(s-\tau)) \, dW_i(s) \qquad (5.4.20)$$

Therefore

$$E \,|\, X(t) - e^{A\tau} X(t-\tau) \,|^2 \le \theta \int_{t-\tau}^{t} e^{-\lambda(t-s)} E \,|\, X(s) - e^{A\tau} X(s-\tau) \,|^2 ds$$

where θ is defined by (5.4.9). It follows that for $t_0 + k\tau \le t \le t_0 + (k+1)\tau$, $k = 1, 2, \ldots,$ we have

$$e^{\lambda t} E \,|\, X(t) - e^{A\tau} X(t-\tau) \,|^2 \le \theta \int_{t_0+(k-1)\tau}^{t_0+k\tau} e^{\lambda s} E \,|\, X(s) - e^{A\tau} X(s-\tau) \,|^2 ds$$

$$+ \theta \int_{t_0+k\tau}^{t} e^{\lambda s} E \,|\, X(s) - e^{A\tau} X(s-\tau) \,|^2 ds \qquad (5.4.21)$$

Applying (5.4.18) and (5.4.15) to this and then using the Gronwall inequality, we deduceimmediatelythat

$$e^{\lambda t} E|X(t) - e^{A\tau} X(t-\tau)|^2 \le 2\theta e^{\lambda(t_0+\tau)} (\tau c_1 + \mu\beta) e^{\theta\tau} := c_2 \tag{5.4.22}$$

for $t_0 + \tau \le t \le t_0 + 2\tau$. By induction, we can easily prove

$$e^{\lambda t} E|X(t) - e^{A\tau} X(t-\tau)|^2 \le c_2 (\theta\tau e^{\theta\tau})^{k-1} \tag{5.4.23}$$

holds for $t_0 + k\tau \le t \le t_0 + (k+1)\tau$, $k = 1, 2, \cdots$. It means (5.4.16) holds by assumption (5.4.8)

Step 3

Combining (5.4.19), (5.4.16), (5.4.18) and (5.4.15) we arrive at

$$E|X(t)|^2 \le (m+2)\mu e^{-\lambda(t-t_0)} E|\varphi(t_0)|^2 +$$

$$+ (m+2)(t+m)\mu K e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} E|X(s) - e^{A\tau} X(s-\tau)|^2 ds$$

$$\le (m+2)\mu e^{-\lambda(t-t_0)} E|\varphi(t_0)|^2 +$$

$$+ (m+2)(t+m)\mu K e^{-\lambda t} \left\{ t c_2 + 2 e^{\lambda(t_0+\tau)} (\tau c_1 + \mu\beta) \right\} \tag{5.4.24}$$

for all $t \ge t_0$. This implies that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log E|X(t)|^2 \le -\lambda$$

which is the desired result (5.4.17) and we complete the proof of Lemma 5.4.3.

We now begin to prove Theorem 5.4.1.

Proof of Theorem 5.4.1. Denote by $X(t, t_0, \varphi)$ $(t \ge t_0)$ the solution of Eq.(5.4.4). For $t \ge 0$, let

$$x(t) = X(t+t_0, t_0, \varphi), \qquad m_i(t) = M_i(t+t_0) - M_i(t_0)$$

$$\beta(t) = \mu(t+t_0) - \mu(t_0), \qquad \mathcal{G}_t = \mathcal{F}_{t+t_0}$$

Then $x(t)$ is the solution of the equation

$$x(t) = \varphi(t_0) + \int_0^t A\, x(s)\, d\beta(s) + \int_0^t B(s)\,(x(s) - e^{A\tau} x(s-\tau))\, d\beta(s) +$$

$$+ \sum_{i=1}^{m} \int_0^t D_i(s)\,(x(s) - e^{A\tau}\,x(s-\tau))\,dm_i(s)\,, \quad t \geq 0 \tag{5.4.25}$$

$$x(t) = \varphi(t+t_o), \qquad -\tau \leq t \leq 0$$

where the stochastic integrals are relative to $\{\mathcal{G}_t\}_{t\geq 0}$. Define $\{\mathcal{G}_t\}$-stopping times

$$T_t := \inf\{ s \geq 0 : \beta(s) > t \}, \quad t \geq 0$$

In view of Theorem 5.2.2, we have

$$x(T_t) = \varphi(t_o) + \int_0^t A\,x(T_s)\,ds + \int_0^t B(T_s)\,(\,x(T_s) - e^{A\tau}\,x(T_s-\tau)\,)\,ds +$$

$$+ \sum_{i=1}^{m} \int_0^t D_i(T_s)\,(x(T_s) - e^{A\tau}\,x(T_s-\tau))\,dm_i(T_s)\,, \quad t \geq 0 \tag{5.4.26}$$

By Lemma 5.4.3 we arrive at

$$e^{\lambda t}\,E\,|\,x(T_t) - e^{A\tau}\,x(T_t-\tau)\,|^2 \leq c_3, \quad t \geq 0 \tag{5.4.27}$$

where c_3 and the following c_i ($i = 4, 5, \ldots$) are all constants. By Theorem 2.4.1, it follows from (5.4.25) that

$$x(t) = e^{A\beta(t)}\,\varphi(t_o) + \int_0^t e^{A[\,\beta(t)-\beta(s)\,]}\,B(s)\,(\,x(s) - e^{A\tau}\,x(s-\tau)\,)\,d\beta(s)$$

$$+ \sum_{i=1}^{m} \int_0^t e^{A[\,\beta(t)-\beta(s)\,]}\,D_i(s)\,(\,x(s) - e^{A\tau}\,x(s-\tau)\,)\,dm_i(s) \tag{5.4.28}$$

An application of Theorem 5.2.2 and Lemma 3.2.1 to this yields

$$x(t) = e^{A\beta(t)}\,\varphi(t_o) + \int_0^{\beta(t)} e^{A[\,\beta(t)-s\,]}\,B(T_s)\,(\,x(T_s) - e^{A\tau}\,x(T_s-\tau)\,)\,ds +$$

$$+\sum_{i=1}^{m}\int_{0}^{\beta(t)} e^{A[\beta(t)-s]} D_i(T_s)\,(\,x(T_s) - e^{A\tau}x(T_s-\tau)\,)\, dm_i(s) \qquad (5.4.29)$$

Consequently

$$E\,|\,x(t)\,|^2 \le c_4\,e^{-\lambda\alpha t} + c_5\,(\,\rho(t)+1)\,e^{-\alpha\lambda t}\int_0^{\rho(t)} e^{\lambda s}\,E\,|\,x(T_s) \;-\; e^{A\tau}x(T_s-\tau)\,|^2\,ds$$

$$\le c_4\,e^{-\lambda\alpha t} + c_6\,(\,\rho(t)+1)\,\rho(t)\,e^{-\alpha\lambda t} \qquad (5.4.30)$$

where the assumptions and (5.4.27) have been used. It follows from (5.4.30) that

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log E\,|\,x(t)\,|^2 \;\le\; -\,\alpha\,\lambda \;+\; 2\,\zeta$$

which means (5.4.13) holds. The proof is complete.

Theorem 5.4.4. Let conditions (1), (2) and (4) in Theorem 5.4.1 hold. Assume
(3)′ τ is such that $\theta\tau\,e^{\theta\tau} > 1$ and
(5)′ there exist $\alpha_i > 0$ and $\delta_i \ge 0$ ($i = 1, 2$) such that

$$\alpha_1 t \,-\, \delta_1 \;\le\; \mu(t+s) - \mu(s) \;\le\; \alpha_2 t \,+\, \delta_2 \qquad \text{for all } t \ge 0 \text{ and } s \ge 0 \qquad (5.4.31)$$

Then

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log E\,|\,X(t)\,|^2 \;\le\; -\,\alpha_1\lambda \,+\, \alpha_2\theta \,+\, \alpha_2\log(\,\theta\tau\,)\,/\,\tau \qquad (5.4.32)$$

Remark 5.4.5. In applications, we can choose τ such that

$$\theta\tau \;=\; K\,(m+1)\,(\tau+m)\,\tau \;\ge\; 0.567144$$

Proof. We use the same notations as the proof of Theorem 5.4.1. In the same way as the proof of (5.4.23) we can prove

$$e^{\lambda t}\,E\,|\,x(T_t) \;-\; e^{A\tau}x(T_t-\tau)\,|^2 \;\le\; c_7\,\gamma^{t/\tau}, \qquad t \ge 0 \qquad (5.4.33)$$

where $\gamma = \theta\tau\,e^{\theta\tau}$. It is similar to the proof of (5.4.30) to prove

$$E\,|\,x(t)\,|^2 \;\le\;\le\; c_8\,e^{-\lambda\alpha_1 t} + c_9\,(\,t+1)\,e^{-\lambda\alpha_1 t}\,\gamma^{\alpha_2 t/\tau}$$

for all $t \ge 0$. This implies

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log E\,|\,x(t)\,|^2\ \le\le\ -\alpha_1\lambda + \alpha_2\theta + \alpha_2\log(\theta\tau)\,/\,\tau$$

which means (5.4.32) holds. The proof is complete.

Example 5.4.6. Let us finally give a simple example to illustrate our results. Consider Eq.(5.4.4) with

$$n = 2,\quad m = 3,\quad |\,B(t)\,|^2 \le 5,\quad |\,D_i\,|^2 \le 5,\quad A = \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}$$

It follows from

$$e^{At} = \frac{1}{5}\begin{bmatrix} 2e^{-t} + 3e^{-2t}, & 2e^{-t} - 2e^{-2t} \\ 3e^{-t} - 3e^{-2t}, & 3e^{-t} + 2e^{-2t} \end{bmatrix}$$

that

$$|\,e^{At}\,|^2 \le 5\,e^{-2t}$$

We assume

$$t - \delta_1 \le \mu(t+s) - \mu(s) \le t + \delta_2 \qquad \text{for all } t \ge 0 \text{ and } s \ge 0$$

where δ_1 and δ_2 are positive constants. If we choose τ so small that

$$\theta\,\tau = K(m+1)(\tau+m)\,\tau = 20(\tau+3)\tau \le 0.567143$$

i.e., $\tau \le 0.00942$ and assume

$$E\int_{t_o-\tau}^{t_o} |\varphi(s)|^2\, d\mu(s+\tau) < \infty$$

then, by Theorem 5.4.1, we have

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log E\,|\,X(t)\,|^2 \le -2$$

If we choose $\tau = 0.0095$, then $\theta\tau = 0.5718 > 0.567144$. Hence, by Theorem 5.4.4, we have

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log E\,|\,X(t)\,|^2 \le -0.6476$$

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